Even Stiefel-Whitney invariants for anti-hermitian quaternionic forms

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Introduction

In [3], Serre defines the notion of cohomological invariant of an algebraic group. More generally, if K is a base field, $\mathrm{Field}_{/K}$ is the category of fields over K, and given functors $F : \mathrm{Field}_{/K} \to \mathrm{Set}$ and $A : \mathrm{Field}_{/K} \to \mathrm{Ab}$, an invariant of F with values in A is simply a natural transformation from F to A, where A is seen as a set-valued invariant by forgetting the group structure. In other words, if $\alpha \in \mathrm{Inv}(F, A)$ is such an invariant, then for any extension L/K and any $x \in F(L)$, it yields an element $\alpha(L) \in A(L)$, and this is compatible with scalar extensions in the sense that if E/L is an over-extension then $\alpha(x_E) = \alpha(x)_E \in A(E)$. We always assume in this article that the base field has characteristic different from 2.

We speak of invariants of an algebraic group G over K when $F(L)$ is the cohomology set $H^1(L, G)$, which can be identified with the set of isomorphism classes of G_L-torsors, and we speak of cohomological invariants when $A(L) = H^d(L, C)$ where C is some Galois module defined over K (we then say that the cohomological invariants have coefficients in C). We also consider Witt invariants, which correspond to $A(L) = W(L)$, the Witt group of L. When G is a classical group, the corresponding functor F usually has some natural algebraic reformulation in terms of bilinear forms or algebras with involution (see [3] and [8]).

When $G = O(A, \sigma)$ where (A, σ) is a central simple algebra with orthogonal involution over K, then $H^1(L, G)$ can be identified with the set of isometry classes of hermitian forms over (A, σ) of reduced dimension $n = \deg(A)$ ([8, 29.26]). Up to isomorphism, this functor only depends on n and the Brauer class $[A] \in \mathrm{Br}(K)$. In particular, if $G = O(V, q)$ where (V, q) is a non-degenerate quadratic space of dimension n, this functor can be identified with Quad_n, the functor of isometry classes of n-dimensional non-degenerate quadratic forms. In this case, Serre gave in [3] a complete description of Witt and cohomological invariants: the Witt invariants are combinations of λ-operations λ^d, and the cohomological invariants are combinations of Stiefel-Whitney invariants w_d.

When A is not split, the cohomological invariants of $O(A, \sigma)$ are only known in cohomological degree up to 3 (see [8] for degree 1 and 2, and [9] for degree 3). When the index of A is 2, the problem amounts to finding cohomological invariants of skew-hermitian forms of reduced dimension n (for some n) over some quaternion algebra Q endowed with its canonical symplectic involution γ. Some progress was made in [1], which uses descent methods from the generic splitting field of Q (the function field of its Sever-Brauer variety, which is somewhat understood as it is the function field of a conic) to extend Stiefel-Whitney invariants to such hermitian forms. Unfortunately, this only yields invariants in $H^d(K,\mu_4)/[Q]\cdot H^{d-2}(K,\mu_2)$ instead of $H^d(K,\mu_2)$, and more importantly the argument is flawed, and actually only works for an invariant which is well-defined on similarity classes instead of isometry classes (what we call here an *even* invariant), in particular for what we call here the even Stiefel-Whitney invariants.

Our main result (Theorem 3.20) is that actually all cohomological invariants of similarity classes of quadratic forms do extend to invariants of skew-hermitian quaternionic forms (not with values in some quotient). Our method is to lift cohomological invariants to Witt invariants, and use the fact that λ-operations are also defined for ε-hermitian forms over algebras with involutions ([5]) to extend those Witt invariants to such hermitian forms. We discuss this general method in Section 3.1. In general it is not enough, as doing this yields constant invariants of a degree lower than expected (if we start from an invariant of degree d, then we generally find a constant invariant of degree d/r where r is the index of the algebra with involution, so $r=2$ for a quaternion algebra). We develop in Section 1 a general framework to manage this kind of situations, and in Section 3.4 we show how to clean up this constant obstruction to actually extend the cohomological invariants to quaternionic forms.

1 Invariants in a filtered group

In this section we study a few generalities regarding invariants with values in $\mathbb{N}$-filtered groups. It would be possible to include more general filtering sets, but at the cost of a few technicalities, and we only need the case of $\mathbb{N}$ for our application.

1.1 Filtered and graded functors

To set up notations, if A is an $\mathbb{N}$-filtered group we write $A_{\geqslant n}\subset A$ for the corresponding subgroup for each $n\in\mathbb{N}$, with $A_{\geqslant n+1}\subset A_{\geqslant n}$ as the notation suggests (we also always assume that $A_{\geqslant 0}=A$), and then the induced $\mathbb{N}$-graded group is $\mathrm{gr}(A)=\bigoplus A_n$ with $A_n=A_{\geqslant n}/A_{\geqslant n+1}$. Any $\mathbb{N}$-graded group $B=\bigoplus B_n$ is also naturally filtered, with $B_{\geqslant n}=\bigoplus_{m\geqslant n}B_m$, and in that case $\mathrm{gr}(B)\simeq B$ canonically.

We define $\mathrm{Ab}_{\mathbb{N}-\mathrm{filt}}$ to be the category of $\mathbb{N}$-filtered abelian group, and $\mathrm{Ab}_{\mathbb{N}-\mathrm{grad}}$ to be the category of $\mathbb{N}$-graded abelian groups. Of course a morphism of filtered groups is a group morphism $f:A\to B$ such that $f(A_{\geqslant n})\subset B_{\geqslant n}$ for all $n\in\mathbb{N}$. The previous construction $A\mapsto\mathrm{gr}(A)$ defines a functor $\mathrm{gr}:\mathrm{Ab}_{\mathbb{N}-\mathrm{filt}}\to\mathrm{Ab}_{\mathbb{N}-\mathrm{grad}}$ in a clear way, and the canonical filtering of a graded group defines a functor $\mathrm{Ab}_{\mathbb{N}-\mathrm{grad}}\to\mathrm{Ab}_{\mathbb{N}-\mathrm{filt}}$ such that the composition is isomorphic to the identity of $\mathrm{Ab}_{\mathbb{N}-\mathrm{grad}}$.

Let $F:\mathrm{Field}_{/K}\to\mathrm{Set}$ and $A:\mathrm{Field}_{/K}\to\mathrm{Ab}_{\mathbb{N}-\mathrm{filt}}$ be two functors. There are obvious subfunctors $A_{\geqslant n}\subset A$ for each $n\in\mathbb{N}$. We write $B=\mathrm{gr}\circ A:\mathrm{Field}_{/K}\to\mathrm{Ab}_{\mathbb{N}-\mathrm{grad}}$. Since $B(L)$ is by definition graded for each L/K, we get groups $B_n(L)$, which define functors $B_n:\mathrm{Field}_{/K}\to\mathrm{Ab}$; by definition, for any extension L/K, we simply have $B(L)=\bigoplus_{n\in\mathbb{N}}B_n(L)$, and $B_n(L)=A_{\geqslant n}(L)/A_{\geqslant n+1}(L)$. Composing B with the canonical $\mathrm{Ab}_{\mathbb{N}-\mathrm{grad}}\to\mathrm{Ab}_{\mathbb{N}-\mathrm{filt}}$, we may also see B as a functor to filtered groups, which just amounts to taking $B_{\geqslant L} = \bigoplus B_m(L)$.

Example 1.1. If $A(L) = W(L)$ is the Witt group of the field L (with the obvious functor structure), endowed with its fundamental filtration $W_{\geqslant n}(L) = I^n(L)$, then $B_n(L) = H^n(L, \mu_2)$ is canonically the mod 2 Galois cohomology of L (this is essentially a reformulation of the Milnor conjecture, see [7] for instance for an exposition of this topic).

1.2 Invariants

Then we are interested in the group of invariants $M = \mathrm{Inv}(F, A)$ and $N = \mathrm{Inv}(F, B)$ of F with values in A and B, that is natural transformations $F \to A$ and $F \to B$, where we see A and B as functors to Set by forgetting the filtered/graded group structure. Clearly, M has a canonical structure of abelian group given by pointwise addition, and even a structure of $\mathbb{N}$-filtered group if we define $M_{\geqslant n}$ to be the image of the natural map $\mathrm{Inv}(F, A_{\geqslant n}) \to \mathrm{Inv}(F, A)$. In fact, $M_{\geqslant n}$ is nothing but the subgroup of invariants $\alpha \in M$ such that for all L/K and all $x \in F(L)$, $\alpha_L(x) \in A_{\geqslant n}(L) \subset A(L)$. The same analysis goes for N, seeing B as a functor to $\mathrm{Ab}_{\mathbb{N}-\mathrm{filt}}$.

Note that even though the filtered structure on B induces a filtered structure on N, this does not work the same way with the graded structure: N is *not* graded in a natural way. In fact, $\mathrm{gr}(N)$ is $\overline{N} = \bigoplus_{n\in\mathbb{N}} N_n$, where of course $N_n = \mathrm{Inv}(F, H_n)$, and the natural inclusion $\overline{N} \subset N$ is in general strict. The invariants in N are *locally* of bounded degree: if $\alpha \in N$ then, for any $x \in F(L)$, there is $m \in \mathbb{N}$ depending on x such that $\alpha_L(x)$ is a combination of elements of $B_n(L)$ for $n \leqslant m$. Then the invariants in $\overline{N}$ are those with *globally* bounded degree. Also note that in general $\overline{N}$ is *not* the graded group induced by M, as the canonical map $M_{\geqslant n} \to N_n$ has no reason to be surjective. (but its kernel is indeed $M_{\geqslant n+1}$, so there is an injective morphism of graded groups $\mathrm{gr}(M) \to \mathrm{gr}(N) = \overline{N}$). In fact:

Definition 1.2. *We say that an invariant $\alpha \in N_n$ is liftable if it lies in the image of $M_{\geqslant n} \to N_n$. The subgroup of liftable invariants is denoted by $\widetilde{N}_n$, and we set $\widetilde{N} = \bigoplus_{n\in\mathbb{N}} \widetilde{N}_n \subset \overline{N}$.*

Then by definition $\widetilde{N}_n \simeq M_{\geqslant n}/M_{\geqslant n+1}$, and $\widetilde{N}$ is canonically the graded group induced by M. To summarize:

$$\widetilde{N} = \mathrm{gr}(M) \subset \overline{N} = \mathrm{gr}(N) \subset N.$$

Example 1.3. If $A = W$ is the Witt group functor, and $F = \mathrm{Quad}_r$ (so $F(L)$ is the set of isometry classes of non-degenerate quadratic forms of dimension r), then all invariants are liftable (see Proposition 2.16). It is also the case when $F = I^d$ (see [4]). This means that in those cases $\overline{N} = \widetilde{N}$.

1.3 Visible and hidden degree

Given an invariant $\alpha \in M$, there is a first naive way to use it to define an invariant $\overline{\alpha} \in \overline{N}$:

Definition 1.4. *Let $\alpha \in M$. We define its visible degree* $\mathrm{vis}(\alpha) \in \mathbb{N} \cup \{\infty\}$ *as*

$$\mathrm{vis}(\alpha) = \sup\{n \in \mathbb{N} \,|\, \alpha \in M_{\geqslant n}\} \in \mathbb{N} \cup \{\infty\}$$

and we define $\overline{\alpha} \in N_{\mathrm{vis}(\alpha)}$ as its image by the canonical map if $\mathrm{vis}(\alpha) \in \mathbb{N}$*, and $\overline{\alpha} = 0 \in N$ if* $\mathrm{vis}(\alpha) = \infty$*.*

Remark 1.5. Saying that $\mathrm{vis}(\alpha) = \infty$ amounts to $\alpha(x) \in \bigcap_{n\in\mathbb{N}} A_{\geqslant n}(L)$ for all L/K and $x \in F(L)$. If for all L/K the filtration on $A(L)$ is complete, meaning that $\bigcap_{n\in\mathbb{N}} A_{\geqslant n}(L) = 0$, then $\mathrm{vis}(\alpha) = \infty$ is equivalent to $\alpha = 0$. This is the case when $A = W$, by the Arason-Pfister Hauptsatz ([2, Cor 23.8]).

The issue is that it is entirely possible that some non-constant $\alpha \in M$ (which, as it is not constant, actually carries information), induces a constant invariant $\overline{\alpha} \in N$. This is actually very simple to set up: take some $\alpha \in M$ with $\mathrm{vis}(\alpha) = n$, and consider $\beta = \alpha + u$ where $u \in A_{pgqn-1}(K) \setminus A_{pgqn}(K)$, seen as a constant invariant. Then even if α is an interesting invariant (and therefore β also carries interesting information), we get $\mathrm{vis}(\beta) = n - 1$ and $\overline{\beta}$ is just the constant invariant $\overline{u} \in N_{n-1}$.

We now explain how to avoid this situation, and always extract non-trivial information in N from a non-constant invariant in M. For any $n \in \mathbb{N}$, we can wee the elements of $A_{\geqslant n}(K)$ as constant invariants in $M_{\geqslant n}$, and thus $A(K)$ is a filtered subgroup of M. Likewise, $B(K)$ is a graded subgroup of $\widetilde{N}$. Let us write $\widehat{M}_{\geqslant n} = M_{\geqslant n}/A_{\geqslant n}(K)$ and $\widehat{N}_n = \widetilde{N}_n/B_n(K)$.

Lemma 1.6. *The short exact sequence*

$$0 \to M_{\geqslant n+1} \to M_{\geqslant n} \to \widetilde{N}_n \to 0$$

induces a short exact sequence

$$0 \to \widehat{M}_{\geqslant n+1} \to \widehat{M}_{\geqslant n} \to \widehat{N}_n \to 0.$$

Proof. The kernel of $M_{\geqslant n+1} \to M_{\geqslant n}/A_{\geqslant n}(K)$ consists of invariants in $M_{\geqslant n+1}$ which are constant, therefore it is exactly $A_{\geqslant n+1}(K)$.

The kernel of $M_{\geqslant n} \to \widetilde{N}_n/B_n(K)$ consists of invariants in $M_{\geqslant n}$ such that they are constant modulo $M_{\geqslant n+1}$, therefore it is $A_{\geqslant n}(K) + M_{\geqslant n+1}$. This establishes the exact sequence. □

This means that if we write $\widehat{M} = M/A(K)$ and $\widehat{N} = N/B(K)$, then $\widehat{M}$ is filtered and $\widehat{N}$ is graded, and $\widehat{N} = \mathrm{gr}(\widehat{M})$.

Definition 1.7. *Let $\alpha \in \widehat{M}$. We define its hidden degree* $\mathrm{hid}(\alpha) \in \mathbb{N} \cup \{\infty\}$ *as*

$$\mathrm{hid}(\alpha) = \sup\{n \in \mathbb{N} \,|\, \alpha \in \widehat{M}_{\geqslant n}\} \in \mathbb{N} \cup \{\infty\}$$

and we define $[\alpha] \in \widehat{N}_{\mathrm{hid}(\alpha)}$ as its image by the canonical map if $\mathrm{hid}(\alpha) \in \mathbb{N}$*, and $[\alpha] = 0 \in \widehat{N}$ if* $\mathrm{hid}(\alpha) = \infty$*.*

If $\alpha \in M$, then $\mathrm{hid}(\alpha)$ *and $[\alpha]$ are defined through the class of α in $\widehat{M}$.*

Proposition 1.8. *Assume that for all L/K the filtration on $A(L)$ is complete. Then for any $\alpha \in M$, there is equivalence between:*

- *α is constant,*
- $\mathrm{hid}(\alpha) = \infty$*,*
- $[\alpha] = 0$*.*

If α is not constant, $\mathrm{hid}(\alpha)$ *is the only $n \in \mathbb{N}$ such that there exists $u \in A(K)$ with $\alpha - u \in M_{\geqslant n}$ and the class of $\alpha - u$ in N_n is not constant. Precisely, if $m < n$ then there are $u \in A(K)$ such that $\alpha - u \in M_{\geqslant m}$, but the class of $\alpha - u$ in N_m is constant; and if $m > n$, then there are no $u \in A(K)$ such that $\alpha - u \in M_{\geqslant m}$.*

Proof. If α is constant, then its class in $\widehat{M}$ is zero, so $\mathrm{hid}(\alpha) = \infty$, and $[\alpha] = 0$.

Assume that α is not constant. Then there exists L/K and $x, y \in F(L)$ such that $\alpha(x) \neq \alpha(y)$. By hypothesis, there is n such that $\alpha(x) - \alpha(y) \notin A_{\geqslant n}(L)$. Then $\alpha \notin M_{\geqslant n}$, and in particular the class of α in $\widehat{M}$ is not in $\widehat{M}_{\geqslant n}$, so $\mathrm{hid}(\alpha) \leqslant n$ is finite.

Let $n = \mathrm{hid}(\alpha)$. By definition, the class of α in $\widehat{M}$ is in $\widehat{M}_{\geqslant n}$ but not in $\widehat{M}_{\geqslant n+1}$. This means that there is some $u \in A(K)$ such that $\alpha - u \in M_{\geqslant n}$, but for any $v \in A(K)$, $\alpha - v \notin M_{\geqslant n+1} + A_{\geqslant n}(K)$. Taking $v = u$, we see that the class of $\alpha - u$ in N_n is not in $B_n(K)$.

Now let $m \neq n$. If $m > n$, we know that for any $v \in A(K)$, $\alpha - v \notin M_{\geqslant n+1}$, so $\alpha - v \notin M_{\geqslant m}$. And if $m < n$, and $v \in A(K)$ is such that $\alpha - v \in M_{\geqslant m}$, then the class of $\alpha - u$ in N_m is 0, so the class of $\alpha - v$ in N_m is also the class of $v - u$ and therefore is constant.

In particular, since $\alpha - u$ is not constant, $[\alpha - u] = [\alpha] \in N_n$ is not zero. $\square$

We can compare the visible and hidden degree:

Proposition 1.9. *Assume again that for all L/K the filtration on $A(L)$ is complete. Let $\alpha \in M$ be non-zero. Then* $\mathrm{hid}(\alpha) \geqslant \mathrm{vis}(\alpha)$*, with equality exactly when $\overline{\alpha}$ is not constant, and in that case $[\alpha]$ is the class of $\overline{\alpha}$ in $\widehat{N}$.*

Proof. Clearly if $\alpha \in M_{\geqslant n}$ then its class in $\widehat{M}$ is in $\widehat{M}_{\geqslant n}$, so $\mathrm{hid}(\alpha) \geqslant \mathrm{vis}(\alpha)$.

If $\overline{\alpha}$ is not constant, then $n = \mathrm{vis}(\alpha)$ satisfies that $\alpha \in M_{\geqslant n}$ and its class in N_n is not constant, so according to Proposition 1.8 we have $n = \mathrm{hid}(\alpha)$.

If $\mathrm{hid}(\alpha) = \mathrm{vis}(\alpha)$, then since $\alpha \neq 0$ this value is finite; let n be this value. Let $u \in A(K)$ be such that $\alpha - u \in M_{\geqslant n}$ and the class of $\alpha - u \in N_n$ is not constant. Since $n = \mathrm{vis}(\alpha)$, $\alpha \in M_{\geqslant n}$, so $u \in A_{\geqslant n}(K)$. Then the class of $\alpha - u$ in N_n is $\overline{\alpha} - \overline{u}$, and since this is not constant, neither is $\overline{\alpha}$. Finally, $[\alpha]$ is then the class of $\overline{\alpha} - \overline{u}$ in $\widehat{N}_n$, which of course is the class of $\overline{\alpha}$. $\square$

2 Invariants of split hermitian forms

For any functor $F : \mathrm{Field}_{/K} \to \mathrm{Set}$, let us write $IW(F)$ and $IC(F)$ respectively for the Witt and cohomological invariants of F, that is $IW(F) = \mathrm{Inv}(F, W)$ and $IC(F) = \mathrm{Inv}(F, H^*(\bullet, \mu_2)$. We also write $IW^{\geqslant d}(F) = \mathrm{Inv}(F, I^d)$ and $IC^d(F) = \mathrm{Inv}(F, H^d(\bullet, \mu_2))$ (recall from Section 1.2 that this defines a filtering of $IW(F)$, but not exactly a grading of $IC(F)$).

Let (A,σ) be an algebra with involution, and let $\mathrm{Herm}_n^\varepsilon(A,\sigma) : \mathrm{Field}_{/K} \to \mathrm{Set}$ be the functor of isometry classes of $\varepsilon-$-hermitian forms of reduced dimension n over (A,σ). Ideally we would like to be able to describe $IW(\mathrm{Herm}_n^\varepsilon(A,\sigma))$ and $IC(\mathrm{Herm}_n^\varepsilon(A,\sigma))$. In this article we provide a certain program to tackle this problem, and give significant advances when A is a quaternion algebra.

But first we take a look at the simplest case: when A is split. We then call ε-hermitian forms over (A,σ) *split hermitian forms.* When the adjoint involutions of ε-hermitian forms over (A,σ) are symplectic (which happens if σ is symplectic and $\varepsilon = 1$, or if σ is orthogonal and $\varepsilon = -1$), then up to isometry there is only one ε-hermitian form of reduced dimension n over (A_L,σ_L) for all L/K, so the only invariants are constant and the problem is very uninteresting (in contrast, when A is not split, $\mathrm{Herm}_n^\varepsilon(A,\sigma)$ can be very interesting and complicated). So for the rest of this section we always assume that either σ is orthognal and $\varepsilon = 1$, or σ is symplectic and $\varepsilon = -1$.

In that case, choosing a hermitian Morita equivalence between (A,σ) and (K,Id) gives a non-canonical isomorphism of funtors $\mathrm{Herm}_n^\varepsilon(A,\sigma) \cong \mathrm{Quad}_n$. Choosing a different Morita equivalence yields the same isomorphism up to multiplying all quadratic forms by $\langle\lambda\rangle$ for some (uniformly chosen) $\lambda \in K^\times$. Therefore we get a canonical map

$$\mathrm{Herm}_n^\varepsilon(A,\sigma) \to \mathrm{Quad}_n / \sim$$

where $\mathrm{Quad}_n / \sim$ is the functor of similarity classes of n-dimensional quadratic forms. This factorizes as

$$\mathrm{Herm}_n^\varepsilon(A,\sigma) \to \mathrm{Herm}_n^\varepsilon(A,\sigma)/\sim \xrightarrow{\sim} \mathrm{Quad}_n / \sim$$

where again we took similarity classes of hermitian forms.

The non-canonical correspondence with quadratic forms shows that the invariants of $\mathrm{Herm}_n^\varepsilon(A,\sigma)$ are the same as those of Quad_n, which are well-understood, but it turns out that because this correspondence is not canonical, we can't hope to extend all invariants of Quad_n in the case where A is not split (there are precise ramification arguments that show that it's not possible). On the other hand, there is hope for the invariants of $\mathrm{Quad}_n / \sim$ (and for Witt invariants at least we can show that they indeed extend).

We know how to describe invariants of $\mathrm{Quad}_n / \sim$, but the combinatorics involved use the fact that forms in Quad_n can be diagonalized, in other words can be decomposed as sums of 1-dimensional forms. When the index of A is $m \in \mathbb{N}^*$, we can decompose forms in $\mathrm{Herm}_n^\varepsilon(A,\sigma)$ as sums of forms of reduced dimension m: if $r \in \mathbb{N}^*$ and X is a finite set with r elements, there is a natural surjective map

$$\left(\mathrm{Herm}_m^\varepsilon(A,\sigma)\right)^X \to \mathrm{Herm}_{rm}^\varepsilon(A,\sigma)$$

given by $(h_i)_{i\in X} \mapsto \sum_{i\in X} h_i$, which also induces

$$\left(\mathrm{Herm}_m^\varepsilon(A,\sigma)\right)^X / \sim \to \mathrm{Herm}_{rm}^\varepsilon(A,\sigma)/\sim$$

where on the left the quotient means that we identify a family $(h_1,\dots,h_r)$ in $\left(\mathrm{Herm}_m^\varepsilon(A,\sigma)\right)^r(L)$ with $(h_1,\dots,h_r)$ if there exists a $\lambda \in L^\times$ independent of i such that $h_i' = \langle\lambda\rangle h_i$.

Thus when A is split we get a commutative diagram:

In this section we describe invariants of those functors, and we try to use a combinatorics that does not use diagonalizations of the m-dimensional forms.

2.1 A coordinate-free approach to Pfister forms

In this article we promote a slightly different (but strictly equivalent) way to view and define Pfister forms, which amounts to doing linear algebra without always choosing a basis.

Traditionally, given elements $a_1, \dots, a_n \in K^\times$, the n-fold Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ is defined as

$$\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \cdot \langle 1, -a_2\rangle \cdots \langle 1, -a_n\rangle.$$

The minus signs have no influence on the definition of the *class* of n-fold Pfister forms (and in early texts they were not present), but they come very handy in the notation when working with quaternion algebras or Galois cohomology.

As a general notation, if X is a set and $(a_i)_{i\in X}$ is a family of elements of $K^\times$, then for any finite subset $I \subset X$ we write

$$\langle a_i\rangle_{i\in I} = \sum_{i\in I} \langle a_i\rangle$$

as well as

$$a_I = \prod_{i\in I} a_i$$

and

$$\langle\langle a_i\rangle\rangle_{i\in I} = \langle\langle a_{i_1}, \dots, a_{i_r}\rangle\rangle$$

if $I = \{i_1, \dots, i_r\}$. We use those notations more generally if the a_i are in $K^\times/(K^\times)$.

Let X be a finite set, and $(a_i)_{i\in X}$ be a family of elements in $K^\times$. By definition,

$$\langle\langle a_i\rangle\rangle_{i\in X} = \langle (-1)^{|I|} a_I\rangle_{I\in\mathcal{P}(X)}.$$

Then notice that in $K^\times/(K^\times)^2$:

$$(-1)^{|I|} a_I \cdot (-1)^{|J|} a_J = (-1)^{|I\Delta J|} a_{I\Delta J}$$

where $I\Delta J$ is the symmetric difference of I and J, which is nothing more than the addition for the standard $\mathbb{F}_2$-vector space structure on $\mathcal{P}(X)$ (notably, it is the addition when viewed in terms of characteristic functions $X \to \{0,1\} \simeq \mathbb{Z}/2\mathbb{Z}$).

This motivates the following:

Definition 2.1. *Let V be a finite-dimensional $\mathbb{F}_2$-vector space, and let $f : V \to K^\times/(K^\times)^2$ be a group morphism (equivalently, a linear map). For any affine subspace $W \subset V$, we set*

$$\langle\langle f|W\rangle\rangle = \langle f(x)\rangle_{x\in W} \in GW(K).$$

Our analysis preceding the definition shows that $\langle\langle f|W\rangle\rangle$ is an n-fold general Pfister form (recall that a general Pfister form is any form that can be written $\langle a\rangle\varphi$ with φ a Pfister form), where $n = \dim(W)$, and that it is a Pfister form if W is a linear subspace. Explicitly, suppose $W = w + W_0$ with $W_0 \subset$ a vector subspace, $(e_i)_{i\in X}$ is an $\mathbb{F}_2$-basis of W_0, $a \in K^\times$ is a representative of $f(w)$, and $-a_i$ a representative of $f(e_i)$ for each $i \in X$. Then

$$\langle\langle f|W\rangle\rangle = \langle a\rangle \langle\langle a_i\rangle\rangle_{i\in X}.$$

We actually need a slightly more general case:

Definition 2.2. *Let q be a quadratic form over K, with similarity factors group $G(q) \subset K^\times$. Let V be a finite-dimensional $\mathbb{F}_2$-vector space, and let $f : V \to K^\times/G(q)$ be a group morphism. Then for any affine subspace $W \subset V$ we define*

$$\langle\langle f|W\rangle\rangle q = \sum_{x\in W} \langle f(x)\rangle q \in GW(K).$$

If $q = \langle\langle f'|W'\rangle\rangle$ for some $f' : V' \to K^\times/(K^\times)^2$ as in Definition 2.1, we write $\langle\langle f|W; f'|W'\rangle\rangle$.

If $a \equiv b mod G(q)$ with $a, b \in K^\times$, then by definition of $G(q)$ we have $\langle a\rangle q = \langle b\rangle q$ in $GW(K)$, so $\langle a\rangle q$ is well-defined for $a \in K^\times/G(q)$, and thus $\langle\langle f|W\rangle\rangle q$ is well-defined. Note that when $q = \langle 1\rangle$ we recover Definition 2.1.

Proposition 2.3. *With the notations of Definition 2.2, if* $\dim(W) = n$ *and q is a general r-fold Pfister form, then $\langle\langle f|W\rangle\rangle q$ is a general $(r+n)$-fold Pfister form.*

If moreover W is a linear subspace and q is a Pfister form, then $\langle\langle f|W\rangle\rangle q$ is a Pfister form.

Proof. Suppose $q = \langle a\rangle\varphi$ for some $a \in K^\times$ and φ an r-fold Pfister form, and let us write $W = w + W_0$ where W_0 is a vector space. Then, recalling that $G(q) = G(\varphi)$:

$$\langle\langle f|W\rangle\rangle q = \sum_{x\in W_0} \langle f(w)\cdot f(x)\rangle \cdot \langle a\rangle\varphi = \langle af(w)\rangle\langle\langle f|W_0\rangle\rangle\varphi$$

so it is enough to prove the second statement, which follows from the same analysis as we did after Definition 2.1. ☐

Remark 2.4. The proof shows that for any q, there exists a (possibly general) Pfister form φ such that $\langle\langle f|W\rangle\rangle q = \varphi q$, but this φ is not well-defined and depends on a choice of basis and a choice of representatives, while $\langle\langle f|W\rangle\rangle q$ does not depend on any choice.

Example 2.5. Let $\varphi = \langle\langle a_i\rangle\rangle_{i\in X}$ be an n-fold Pfister form. As we saw, we can naturally write it as $\varphi = \langle\langle f|\mathcal{P}(X)\rangle\rangle$ by setting $f(\{i\}) = -a_i$. Then $\mathcal{P}_0(X)$ is a hyperplane of $\mathcal{P}(X)$, so we get an $(|X|-1)$-fold Pfister form $\langle\langle f|\mathcal{P}_0(X)\rangle\rangle$, which we call the even part of φ. (Technically this does not depend simply on φ but also on its representation as $\langle\langle a_i\rangle\rangle_{i\in X}$.)

If $X = \{1,\dots,n\}$, we can write it as for instance $\langle\langle a_1a_2, a_1a_3, \dots, a_1a_n\rangle\rangle$, or $\langle\langle a_1a_2, a_2a_3, \dots, a_{n-1}a_n\rangle\rangle$. The various ways to represent it as a Pfister form with the usual notation, correspond to different choices of basis of $\mathcal{P}_0(X)$, and there is no obvious natural choice.

2.2 Even and odd invariants

We now describe a general framework to handle invariants of similarity classes. It has some overlap with [4], but the formalism used there is a little heavy-handed for our purposes, so we give a self-contained account.

Let $F : \mathrm{Field}_{/K} \to \mathrm{Set}$ be a functor. The square classes functor $G : L \mapsto L^\times/(L\times)^2$ is a group functor, and an G-action on F is just an action of $G(L)$ on

$F(L)$ for every L, which is compatible in the obvious way with scalar extensions. In that case we can consider the quotient functor F/G (such that $(F/G)(L) = F(L)/G(L)$ is the usual set-theoretic quotient for the given action).

Example 2.6. Or course our main examples are $F = \mathrm{Quad}_m$, or more generally $(\mathrm{Quad}_r)^r$, as well as $\mathrm{Herm}_n^\varepsilon(A, \sigma)$, each time with the natural action through multiplication by a 1-dimensional form. The quotients are then the corresponding similarity classes functors.

For the rest of this section we fix some $F : \mathrm{Field}_{/K} \to \mathrm{Set}$ with a G-action.

Lemma 2.7. *The canonical map $IW(F/G) \to IW(F)$ (resp. $IC(F/G) \to IC(F)$) is an injective morphism of $W(K)$-algebras (resp. $h^*(K)$-algebras), and its image is the subalgebra of invariants α such that for all L/K, $x \in F(L)$ and $\lambda \in G(L)$, we have $\alpha(\lambda \cdot x) = \alpha(x)$.*

Proof. The fact that $F \to F/G$ is surjective by definition implies that $IW(F/G) \to IW(F)$ and $IC(F/G) \to IC(F)$ are injective. The map on the level of invariants induced by some $F \to F'$ is always an algebra morphism, as the algebra structure is defined pointwise. Finally, α is in the image of the map if and only if it factorizes through an invariant of F/G, which is equivalent to the condition in the statement. $\square$

Definition 2.8. *We say that $\alpha \in IW(F)$ (resp. $\alpha \in IC(F)$) is* even *if it is in the image of $IW(F/G) \to IW(F)$ (resp. $IC(F/G) \to IC(F)$). The subalgebra of even invariants is denoted $IW(F)_0$ (resp. $IC(F)_0$).*

We say that $\alpha \in IW(F)$ is odd *if for all L/K, $x \in F(L)$ and $\lambda \in G(L)$, we have $\alpha(\lambda \cdot x) = \langle\lambda\rangle\alpha(x)$). We write $IW(F)_1$ for the set of odd invariants in $IW(F)$.*

Example 2.9. When $F = \mathrm{Quad}_n$, Serre proved ([3]) that $IW(F)$ is a free $W(K)$-module with basis $(\lambda^d)_{0\leqslant d\leqslant n}$. Then the even invariants are those that are combinations of the λ^d with d even, and the odd invariants are those that are combinations of the λ^d with d odd.

Note that there is no notion of odd cohomological invariant.

Proposition 2.10. *$IW(F)_1$ is a $W(K)$-submodule of $IW(F)$, and actually*

$$IW(F) = IW(F)_0 \oplus IW(F)_1,$$

which turns $IW(F)$ into a $\mathbb{Z}/2\mathbb{Z}$-graded algebra over $W(K)$.

Proof. The fact that $IW(F)_1$ is a submodule is clear by definition. Let $\alpha \in IW(F)$, and let us show that $\alpha = \alpha_0 + \alpha_1$ for some uniquely defined α_0 and α_1, respectively even and odd.

Consider some extension L/K and some $x \in F(L)$. Then we can define a Witt invariant of G over L by $\beta : \lambda \mapsto \alpha(\lambda \cdot x_E)$ for all extensions E/L and $\lambda \in G(E)$. According to [3], there are uniquely defined elements $u, v \in W(L)$ such that $\beta(\lambda) = u + \langle\lambda\rangle v$ for all $\lambda \in G(E)$ for all E/L. Then we can define $\alpha_0(x) = u$ and $\alpha_1(x) = v$, and the uniqueness property of u and v shows that this actually defines invariants $\alpha_0, \alpha_1 \in IW(F)$, and it is clear by construction that they are respectively even and odd. Furthermore, the uniqueness property of u

and v shows that we must have $\alpha_0(x) = u$ and $\alpha_1(x) = v$ for any decomposition $\alpha = \alpha_0 + \alpha_1$.

The fact that this gives a $\mathbb{Z}/2Z$-graded algebra just amounts to checking that the product of two even/odd invariants is even/odd with the usual rules, and that follows directly from the definition. $\square$

Proposition 2.11. *For any $\alpha \in IF(F)$ there is a unique $\widetilde{\alpha} \in IC(F)_0$ such that for all L/K, $x \in F(L)$ and $\lambda \in G(L)$, we have*

$$\alpha(\lambda \cdot x) = \alpha(x) + (\lambda) \cup \widetilde{\alpha}(x),$$

and α is even if and only if $\widetilde{\alpha} = 0$.

Proof. Consider some extension L/K and some $x \in F(L)$. Then we can define a cohomological invariant of G over L by $\beta : \lambda \mapsto \alpha(\lambda \cdot x_E)$ for all extensions E/L and $\lambda \in G(E)$. According to [3], there are uniquely defined elements $u, v \in h^*(L)$ such that $\beta(\lambda) = u + (\lambda) \cup v$ for all $\lambda \in G(E)$ for all E/L. Taking $\lambda = 1$, we get $u = \alpha(x)$. Then we can define $\widetilde{\alpha}(x) = v$, and the uniqueness property of u and v shows that this actually defines an invariant $\widetilde{\alpha} \in IC(F)$, which by definition satisfies the expected formula, and is the only possible one by uniqueness of v.

It is clear by definition that α is even if and only if $\widetilde{\alpha} = 0$. It just remains to show that $\widetilde{\alpha}$ is always even. Now consider the following formula:

$$\begin{aligned}
\alpha((\lambda\mu) \cdot x) &= \alpha(\mu \cdot x) + (\lambda) \cup \widetilde{\alpha}(\mu \cdot x) \\
&= \alpha(x) + (\lambda\mu) \cup \widetilde{\alpha}(x) \\
&= \alpha(x) + (\lambda) \cup \widetilde{\alpha}(x) + (\mu) \cup \widetilde{\alpha}(x).
\end{aligned}$$

As this is valid for all $\lambda, \mu \in G(L)$ for all extensions L, we can take residues at λ to get $\widetilde{\alpha}(\mu \cdot x) = \widetilde{\alpha}(x)$, which shows that $\widetilde{\alpha}$ is even. $\square$

We now consider degrees of invariants, and the connexion between even Witt and cohomological invariants:

Proposition 2.12. *Let $\beta \in IW^{\geqslant d}(F)$. Then its even and odd parts β_0 and β_1 are in $IW^{\geqslant d-1}(F)$, and have the same class in $IC^{d-1}(F)$. If β is even or odd, then its class in $IC^d(F)$ is even.*

In fact, if $\alpha \in IC^d(F)$, then $\widetilde{alpha} \in IC^{d-1}(F)$, and if moreover α is liftable and $\beta \in IW^{\geqslant d}(F)$ is a lift of α, then $\widetilde{\alpha}$ is also liftable and $\beta_0, \beta_1 \in IW^{\geqslant d-1}(F)$ are both lifts of $\widetilde{\alpha}$.

Proof. Let $\beta \in IW^{\geqslant d}(F)$. It follows from the definition of even/odd invariants that for any L/K, any $x \in F(L)$ and $\lambda \in G(L)$:

$$\beta(\lambda \cdot x) = \beta(x) - \langle\langle \lambda \rangle\rangle \beta_1(x). \tag{1}$$

In particular, this means that $\langle\langle \lambda \rangle\rangle \beta_1(x) \in I^d(L)$. Since this is true for all λ over all extensions of L, we can take residues at λ to get that $\beta_1(x) \in I^{d-1}(L)$ (as residue maps satisfy $\partial(I^d) \subset I^{d-1}$, see [3]). This means that $\beta_1 \in IW^{\geqslant d-1}$

Now since $\beta_0 + \beta_1 = \beta \in IW^{\geqslant d}$, we can conclude that $\beta_0 \equiv -\beta_1 \mod IW^{\geqslant d}(F)$. As $IC^d(F)$ is a 2-torsion group, this means that $\beta_0 \equiv \beta_1 \mod IW^{\geqslant d}(F)$, and in particular $\beta_0 \in IW^{\geqslant d-1}(F)$ also.

If β is even or odd, then for all L/K, $x \in F(L)$ and $\lambda \in G(L)$: $\beta(\lambda \cdot x) \equiv \beta(x) \mod I^{d+1}(L)$. This is clear if β is even (we have equality), and when β is odd we use that for any $q \in I^d(L)$, $\langle\lambda\rangle q \equiv q \mod I^{d+1}(L)$.

If $\alpha \in IC^d(F)$, then $\alpha(\lambda \cdot x) = \alpha(x) + (\lambda) \cup \widetilde{\alpha}(x)$ shows that $(\lambda) \cup \widetilde{\alpha}(x)$ is in $h^d(L)$ for all L/K, $x \in F(x)$ and $\lambda \in G(L)$, and taking residues at λ shows that $\widetilde{\alpha(x)}$ is in $h^{d-1}(L)$.

Now if we assume that α is the class of β in $IC^d(F)$, then formula (1) shows that $-\beta_1$ is a lift of $\widetilde{\alpha}$ (and therefore also β_1). Since β_0 and β_1 have the same class in $IC^{d-1}(F)$, they are both lifts. □

Remark 2.13. It would be tempting to guess that any liftable even invariant in $IC(F)_0$ has a lift in $IW(F)_0$, but there is actually no guarantee that this happens. On the other hand, the even invariants that are of the form $\widetilde{\alpha}$ for some liftable α do indeed have an even lift, according to the proposition.

Even assuming that all invariants are liftable, it may very well happen that not all even cohomological invariants have the form $\widetilde{\alpha}$ (a counter-example is given by $F = I^n$ with $n \geqslant 2$, see [4]).

On the other hand, if $F = \mathrm{Quad}_{2r}$ for $r \in \mathbb{N}^*$, then all even cohomological invariants do have the form $\widetilde{\alpha}$ ([4]), and thus they all have a lift in $IW(F)_0$.

2.3 Invariants of quadratic forms

Serre gave in [3] a complete description of the Witt and cohomological invariants of Quad_r. Precisely, $IW(\mathrm{Quad}_r)$ is a free $W(K)$-module with basis $(\lambda^d)_{0\leqslant d\leqslant r}$, where we recall that $\lambda^d : GW(K) \to GW(K)$ is an operation such that

$$\lambda^d(\langle a_i\rangle_{i\in X}) = \langle a_I\rangle_{|I|=d}.$$

These operations turn $GW(K)$ into a pre-λ-ring. Let us give another basis of $IW(\mathrm{Quad}_r)$. In any pre-λ-ring R, one may define

$$P_r^d = \sum_{k=0}^{d} (-1)^k \binom{r-k}{d-k} \lambda^k \tag{2}$$

for all $r \in \mathbb{N}^*$ and $d \in \{0, \dots, r\}$. Clearly, since they form a triangular family with respect to the basis (λ^d), the family $(P_r^d)_{0\leqslant d\leqslant r}$ is also a $W(K)$-basis of $IW(\mathrm{Quad}_r)$.

Then according to [4], a purely formal property of the P_r^d, valid in any pre-λ-ring R, is that

$$P_{s+t}^d(q_1 + q_2) = \sum_{d_1+d_2=d} P_s^{d_1}(q_1) P_t^{d_2}(q_2). \tag{3}$$

for any $q_1, q_2 \in R$.

Proposition 2.14. *Let X be a finite set with r elements, and let $(a_i)_{i\in X} \in (K^\times)^X$ be a family of elements in $K^\times$. Let us define $f : \mathcal{P}(X) \to K^\times/(K^\times)^2$ by $f(\{i\}) = -a_i$. Then*

$$P_r^d(\langle a_i\rangle_{i\in X}) = \sum_{|I|=d} \langle\langle f|\mathcal{P}(I)\rangle\rangle.$$

In particular, $P_r^d \in IW^{\geqslant d}(\mathrm{Quad}_r)$.

Proof. With a simple induction using (3), we see that

$$P_r^d(\langle a_i\rangle_{i\in X}) = \sum_{|I|=d}\prod_{i\in I} P_1^1(\langle a_i\rangle),$$

and we conclude using that $P_1^1(\langle a\rangle) = \langle\langle a\rangle\rangle$, as (2) tells us that $P_1^1 = \lambda^0 - \lambda^1$.

Since $\langle\langle f|\mathcal{P}(I)\rangle\rangle$ is a d-fold Pfister form, it follows that P_r^d has values in I^d. □

When $d = r$, we get a single Pfister form:

$$P_r^r(\langle a_i\rangle_{i\in X}) = \langle\langle f|\mathcal{P}(X)\rangle\rangle = \langle\langle a_i\rangle\rangle_{i\in X}.$$

Remark 2.15. The proposition shows that when $d > r$, P_r^d is zero on Quad_r.

Now regarding cohomological invariants, [3] proves that $IC(\mathrm{Quad}_r)$ is also a free $h^*(K)$-module, with basis $(w_d)_{0\leqslant d\leqslant r}$, where the w_d are the Stiefel-Whitney invariants:

$$w_d(\langle a_i\rangle_{i\in X}) = \sum_{|I|=d} (a_i)_{i\in I} \in h^d(K) \tag{4}$$

where $(a_i)_{i\in I}$ means the cup-product of all $(a_i) \in h^1(K)$ for $i \in I$.

We record the following obvious but important observation:

Proposition 2.16. *The invariant $w_d \in IC(\mathrm{Quad}_r)$ is the class in $IC^d(\mathrm{Quad}_r)$ of $P_r^d \in IW^{\geqslant d}(\mathrm{Quad}_r)$. In particular, all invariants of $IC(\mathrm{Quad}_r)$ are liftable.*

Proof. Comparing (4) and Proposition 2.14, we may conclude using the fact that if $|I| = d$, the class of $\langle\langle a_i\rangle\rangle_{i\in I}$ in $h^d(K)$ is exactly $(a_i)_{i\in I}$. □

2.4 Invariants of families of quadratic forms

As we explained at the beginning of the section, we are also interested in invariants of $(\mathrm{Quad}_m)^X$ where $m \in \mathbb{N}^*$ and X is a finite set with r elements.

For any I-indexed family $(\alpha_d)_{d\in I}$ of invariants in $IW(\mathrm{Quad}_m)$, and any function $\gamma : X \to I$, we may define $\alpha_\gamma \in IW((\mathrm{Quad}_m)^X)$ by

$$\alpha_\gamma((q_i)_{i\in X}) = \prod_{i\in X} \alpha_{\gamma(i)}(q_i). \tag{5}$$

Likewise, if the α_d are in $IC(\mathrm{Quad}_m)$, we get invariants α_γ in $IC((\mathrm{Quad}_m)^X)$ by taking cup-products.

For any $f : X \to \mathbb{N}$, we write $|f| = \sum_{i\in X}\gamma(i)$. Then clearly, if $\alpha_d \in IW^{\geqslant f(d)}(\mathrm{Quad}_m)$ for some function $f : I \to \mathbb{N}$, then $\alpha_\gamma \in IW^{\geqslant |f\circ\gamma|}((\mathrm{Quad}_m)^X)$. The same thing holds for degrees of cohomological invariants.

Also, if β_d is the class of α_d in $IC^{f(d)}(\mathrm{Quad}_m)$ for all $d \in I$, then β_γ is the class of α_γ in $IC^{|f\circ\gamma|}((\mathrm{Quad}_m)^X)$.

Lemma 2.17. *Let $(\alpha_d)_{d\in I}$ be a $W(K)$-basis of $IW(\mathrm{Quad}_r)$, such that it remains a basis when extending the scalars to any extension L/K. Then the α_γ with $\gamma : X \to I$ form a basis of $IW((\mathrm{Quad}_m)^X)$. The same result holds for a basis of $IC(\mathrm{Quad}_r)$ which remains a basis over any extension.*

Proof. We proceed by induction on $|X|$; if $|X| = 1$ then this is clear. Suppose $X = \{x\} \cup Y$, and let $\alpha \in IW((\mathrm{Quad}_m)^X)$. For any L/K and $q \in \mathrm{Quad}_m(L)$, we can define an invariant $\beta_q \in IW((\mathrm{Quad}_m)^Y)$ over L as follows: if E/L is an extension and $(q_i)_{i\in Y} \in \mathrm{Quad}_m(E)^Y$, then let $q_x = q_E \in \mathrm{Quad}_m(E)$, and set $\beta_q((q_i)_{i\in Y}) = \alpha((q_i)_{i\in X}$.

Then β_q decomposes uniquely as $\beta_q = \sum_\gamma a_\gamma(q)\alpha_\gamma$ with $\gamma : Y \to I$ and $a_\gamma(q) \in W(L)$ (we can apply the induction hypothesis over the base field L since the α_d remain a basis over L and any extension). Then for each γ, $q \mapsto a_\gamma(q)$ defines an invariant in $IW(\mathrm{Quad}_m)$ over K, which itself decomposes uniquely as a combination of the α_d with coefficients in $W(K)$. Finally, this means that α itself decomposes uniquely as a combination of the α_γ with $\gamma : X \to I$.

The proof is the same for cohomological invariants. $\square$

We then apply this construction to our basic invariants P_m^d and w_d to get invariants $P_m^\gamma \in IW^{\geqslant|\gamma|}((\mathrm{Quad}_m)^X)$ and $w_\gamma \in IC^{|\gamma|}((\mathrm{Quad}_m)^X)$ for any $\gamma : X \to \{0, \dots, m\}$.

Explicitly:

$$P_m^\gamma((q_i)_{i\in X}) = \prod_{i\in X} P_m^{\gamma(i)}(q_i). \tag{6}$$

and

$$w_\gamma((q_i)_{i\in X}) = \bigcup_{i\in X} w_{\gamma(i)}(q_i). \tag{7}$$

Proposition 2.18. *Let $I = \{0, \dots, m\}$. Then the $W(K)$-module $IW((\mathrm{Quad}_m)^X)$ is free with basis the P_m^γ with $\gamma : X \to I$. Likewise, $IC((\mathrm{Quad}_m)^X)$ is free with basis the w_γ, and w_γ is the class of P_m^γ in $IC^{|\gamma|}((\mathrm{Quad}_m)^X)$. In particular, all invariants in $IC((\mathrm{Quad}_m)^X)$ are liftable.*

Proof. We can apply Lemma 2.17 since the P_m^d form a basis of invariants, and that remains valid over any field extension (as we made no assumption on the base field). The same applies to w_γ. $\square$

Proposition 2.19. *We have for any L/K and any $(q_i)_{i\in X} \in (\mathrm{Quad}_m(L))^X$:*

$$P_{rm}^d(\sum_{i\in X} q_i) = \sum_{|\gamma|=d} P_m^\gamma((q_i)_{i\in X})$$
$$w_d(\sum_{i\in X} q_i) = \sum_{|\gamma|=d} w_\gamma((q_i)_{i\in X}).$$

In other words, the image of P_{rm}^d through the canonical map $IW(\mathrm{Quad}_{rm}) \to IW((\mathrm{Quad}_m)^X)$ is $\sum_{|\gamma|=d} P_m^\gamma$, and likewise the image of w_d in $IC((\mathrm{Quad}_m)^X)$ is $\sum_{|\gamma|=d} w_\gamma((q_i)_{i\in X})$.

Proof. The computation of $P_m^d(\sum_{i\in X} q_i)$ is by induction on the formula (3), and that of $w_d(\sum_{i\in X} q_i)$ follows. $\square$

2.5 Invariants of similarity classes

We now turn to invariants of $Quad_n/\sim$ and $(\mathrm{Quad}_m)^X/\sim$. According to Lemma 2.7, they are identified with the subalgebras of even invariants of Quad_r and $(\mathrm{Quad}_m)^X$ respectively.

We also know that for Witt invariants, the even parts of a system of generators of $IW(F)$ form a system of generators of $IW(F)_0$ for any functor F. This leads to:

Definition 2.20. *We define $Q_n^d \in IW(\mathrm{Quad}/\sim)$ as the even part of $P_n^d \in IW(\mathrm{Quad}_n)$, and $Q_m^\gamma \in IW((\mathrm{Quad}_m)^X/\sim)$ as the even part of $P_m^\gamma \in IW((\mathrm{Quad}_m)^X)$, for any $\gamma : X \to \{0, \dots, m\}$.*

Remark 2.21. Be aware that Q_m^γ is *not* obtained from the Q_m^d through the process described in (5).

Lemma 2.22. *With the same hypotheses as in Lemma 2.17, let us furthermore assume that there is a function $f : I \to \mathbb{Z}/2\mathbb{Z}$ such that $\alpha_d \in IW(\mathrm{Quad}_m)_{f(d)}$ for all $d \in I$ (so the α_d form a graded basis for the $\mathbb{Z}/2\mathbb{Z}$-grading on $IW(\mathrm{Quad}_m)$).*

For each $\gamma : X \to I$ we write $\pi(\gamma) = \sum_{i \in X} f(\gamma(i)) \in \mathbb{Z}/2\mathbb{Z}$. Then the α_γ are a graded basis of $IW((\mathrm{Quad}_m)^X)$ in the sense that $\alpha_\gamma \in IW((\mathrm{Quad}_m)^X)_{\pi(\gamma)}$. In particular, the α_γ with $\pi(\gamma) = 0$ are a basis of $IW((\mathrm{Quad}_m)^X)_0$.

Proof. We already know from Lemma 2.17 that the α_γ form a basis, and the parity of α_γ is immediate from the fact that $IW((\mathrm{Quad}_m)^X)$ is a $Z/2\mathbb{Z}$-graded algebra, and the definition of α_γ as a product. □

Proposition 2.23. *We have $Q_n^d \in IW^{\geqslant d-1}(\mathrm{Quad}_n/\sim)$, and $IW(\mathrm{Quad}_n/\sim)$ is free with basis the Q_n^d with $d \in \{0, \dots, n\}$ which is even. Moreover, all invariants in $IC(\mathrm{Quad}_n/\sim)$ are liftable.*

Let $I = \{0, \dots, m\}$. For any $\gamma : X \to I$, $Q_m^\gamma \in IW^{\geqslant |\gamma|-1}((\mathrm{Quad}_m)^X/\sim)$, and $IW((\mathrm{Quad}_m)^X/\sim)$ is free with basis the Q_m^γ with $\gamma : X \to I$ such that $|\gamma|$ is even.

Proof. The fact that Q_n^d is in $IW^{\geqslant d-1}$ is a direct consequence of Proposition 2.12, as $P_n^d \in IW^{\geqslant d}$. Since the λ^d are even/odd according to the parity of d, the λ^d with d even form a basis of $IW(\mathrm{Quad}_n/\sim)$. Now P_n^d is the sum of λ^d plus combinations of λ^k with $k < d$, so when d is even λ^d is also the leading term of Q_n^d, which means that the family $(Q_n^d)_{d \text{ even}}$ is triangular in the basis $(\lambda^d)_{d \text{ even}}$, so it is also a basis.

The fact that invariants in $IC(\mathrm{Quad}_n/\sim)$ are liftable follows from the discussion in Remark 2.13: all invariants in $IC(\mathrm{Quad}_n)$ are liftable, so it is enough to show that all even invariants in $IC(\mathrm{Quad}_n)$ are of the form $\widetilde{\alpha}$ for some α. This follows from explicit basis computations in [4]: there is a basis v_d of $IC(\mathrm{Quad}_n)$ such that $IC(\mathrm{Quad}_n/\sim)$ is the submodule generated by the v_d with d odd [4, Rem 9.13], and we have $\widetilde{v_{d+1}} = v_d$ when d is odd [4, Prop 7.6].

To show the statement regarding the Q_n^γ, we cannot directly apply Lemma 2.22 to $\alpha_d = P_n^d$, as they are neither even nor odd, but we rather apply it to $\alpha_d = \lambda^d$. Then again, for the same reason as for the Q_n^d, the Q_n^γ with $|\gamma|$ even form a triangular family in the basis $(\alpha_\gamma)_{|\gamma| \text{ even}}$. □

Proposition 2.24. *We have for any L/K and any $(q_i)_{i \in X} \in (\mathrm{Quad}_m(L))^X$:*

$$Q_{rm}^d\Big(\sum_{i \in X} q_i\Big) = \sum_{|\gamma| = d} Q_m^\gamma((q_i)_{i \in X}).$$

In other words, the image of Q_{rm}^d through the canonical map $IW(\mathrm{Quad}_{rm}/\sim) \to IW((\mathrm{Quad}_m)^X/\sim)$ is $\sum_{|\gamma|=d} Q_m^\gamma$.

Proof. This is just taking the even part on both sides of Proposition 2.19. □

2.6 The splitting process

We now have basic Witt invariants P^d_{rm}, Q^d_m, P^γ_{rm} and Q^γ_m of our respective functors Quad_{rm}, $\mathrm{Quad}_{rm}/\sim$, $(\mathrm{Quad}_m)^X$ and $(\mathrm{Quad}_m)^X/\sim$. We know that they have values in I^k for certain $k \in \mathbb{N}$ (respectively, $k = d$, $k = d-1$, $k = |\gamma|$ and $k = |\gamma|-1$), which means that their values can be written as sums of k-fold Pfister forms in each case, and we would like to understand what these Pfister forms look like given the combinatorics of our quadratic forms.

As we explained, the idea of studying invariants of $(\mathrm{Quad}_m)^X$ and $(\mathrm{Quad}_m)^X/\sim$ is that they are stepping stones to the case of hermitian forms over algebras whose index divides m. Now suppose we are studying hermitian forms over such algebras, but we want to understand how to specialize to algebras that actually have a smaller index, say dividing m' with $m' = rm$. Then each m-dimensional form can be further decomposed as a sum of m'-dimensional forms indexed by some set Y with r elements. The case where $m' = 1$ corresponds to working with split algebras, and fully diagonalizing each quadratic form. In general, the situation can be modeled with the canonical morphism

$$(\mathrm{Quad}_m)^{X\times Y} \to (\mathrm{Quad}_{m|Y|})^X$$

for some finite sets X and Y, which is defined by

$$(q_{i,j})_{i\in X, j_inY} \mapsto (\sum_{j\in Y} q_{i,j})_{i\in X}.$$

This induces an injection

$$IW((\mathrm{Quad}_{m|Y|})^X) \to IW((\mathrm{Quad}_m)^{X\times Y}),$$

and one can ask how the basic invariant decompose. For any $\omega : X \times Y \to \mathbb{N}$, we define $\omega_X : X \to \mathbb{N}$ as

$$\omega_X(i) = \sum_{j\in Y} \omega(i,j).$$

It satisfies $|\omega_X| = |\omega|$.

Proposition 2.25. *Let $\gamma : X \to \mathbb{N}$. The image of $P^\gamma_{m|Y|}$ in $IW((\mathrm{Quad}_m)^{X\times Y})$ is*

$$\sum_{\omega : X\times Y \to \mathbb{N}, \omega_X = \gamma} P^\omega_m.$$

Proof. Let $q_{i,j} \in \mathrm{Quad}_m(L)$ for every $(i,j) \in X \times Y$, for some L/K. Then

$$\begin{aligned}
P^\gamma_{m|Y|}((\sum_{j\in Y} q_{i,j})_{i\in X}) &= \prod_{i\in X} P^{\gamma(i)}_m(\sum_{j\in Y} q_{i,j}) \\
&= \prod_{i\in X} \sum_{f:Y\to\mathbb{N}, |f|=\gamma(i)} P^f_m((q_{i,j})_{j\in Y}) \\
&= \sum_{F:X\to(Y\to\mathbb{N}), |F(i,-)|=\gamma(i)} \prod_{i\in X}\prod_{i\in X} P^{F(i)}_m((q_{i,j})_{j\in Y}) \\
&= \sum_{\omega:X\times Y\to\mathbb{N}, \omega_X=\gamma} P^\omega_n((q_{i,j})_{(i,j)\in X\times Y})
\end{aligned}$$

where we used Proposition 2.19 and the correspondence between functions $X \to (Y \to \mathbb{N})$ and $X \times Y \to \mathbb{N}$. $\square$

Note that only the γ with image in $\{0,\dots,m|Y|\}$ yield a non-zero $P^{\gamma}_{m|Y|}$, and likewise only the ω with image in $\{0,\dots,m\}$ give non-zero contribution in the formula.

Corollary 2.26. *Let $\gamma : X \to \mathbb{N}$. The image of $Q^{\gamma}_{m|Y|}$ in $IW((\mathrm{Quad}_m)^{X\times Y}/\sim)$ is*

$$\sum_{x:X\times Y\to\mathbb{N},\omega_X=\gamma} Q^{\omega}_m.$$

Proof. This is just taking the even part on both sides of Proposition 2.25. □

When we take $m = 1$, which means that we fully diagonalize the forms, there is a nice description of the Pfister forms that intervene. Note that in that case, our functions $\gamma : X \to \mathbb{N}$ that are involved in invariants of $(\mathrm{Quad}_1)^X$ only take values in $\{0,1\}$, so we can identify them with subsets $I \subset X$, using the characteristic function χ_I of I. Note that $|\chi_I| = |I|$.

Lemma 2.27. *Let X be a finite set, and let $I \subset X$. Let $(a_i)_{i\in X} \in (K^\times)^X$. We define a morphism $f : \mathcal{P}(X) \to K^\times/(K^\times)^2$ by $f(\{i\}) = -a_i$. Then*

$$P_1^{\chi_I}((\langle a_i\rangle)_{i\in X}) = \langle\langle f|\mathcal{P}(I)\rangle\rangle$$
$$Q_1^{\chi_I}((\langle a_i\rangle)_{i\in X}) = \langle\langle f|\mathcal{P}_0(I)\rangle\rangle.$$

Proof. The first equality simply comes from the fact that $P_1^0(\langle a\rangle)) = 1$ and $P_1^1(\langle a\rangle) = \langle\langle a\rangle\rangle$, so $P_1^{\chi_I}((\langle a_i\rangle)_{i\in X}) = \prod_{i\in I}\langle\langle a_i\rangle\rangle$.

For the second one, notice that if $\lambda\cdot f$ denotes the morphism $\mathcal{P}(X) \to K^\times/(K^\times)^2$ defined by $(\lambda\cdot f)(\{i\}) = \lambda f(\{i\})$, then $(\lambda\cdot f)(I) = f(I)$ if $|I|$ is even, and $(\lambda\cdot f)(I) = \lambda f(I)$ if $|I|$ is odd. So

$$P_1^{\chi_I}((\langle \lambda a_i\rangle)_{i\in X}) = \langle\langle f|\mathcal{P}_0(I)\rangle\rangle + \langle\lambda\rangle\langle\langle f|\mathcal{P}_1(I)\rangle\rangle.$$

Then by definition of the even part of a Witt invariant we get the formula for $Q_1^{\chi_I}$. □

Now if we want to apply that to $(\mathrm{Quad}_1)^{X\times Y} \to (\mathrm{Quad}_{|Y|})^X$, we notice that when $\omega : X\times Y \to \mathbb{N}$ is χ_I for some $I \subset X\times Y$, then $\omega_X : X \to \mathbb{N}$ is the function θ_I defined by

$$\theta_I(i) = |\pi^{-1}(\{i\}) \cap I|$$

where $\pi : X\times Y \to X$ is the canonical projection. In particular, $|\theta_I| = |I|$.

Proposition 2.28. *Let $(q_i)_{i\in X} \in (\mathrm{Quad}_{|Y|}(K))^X$, and let $q = \sum_{i\in X} q_i$. For each $i \in X$, we diagonalize q_i as $\langle a_{i,j}\rangle_{j\in Y}$. Let us define $f : \mathcal{P}(X\times Y) \to K^\times/(K^\times)^2$ by $f(\{(i,j)\}) = -a_{i,j}$. Then:*

$$P^d_{|X||Y|}(q) = \sum_{|I|=d}\langle\langle f|\mathcal{P}(I)\rangle\rangle$$
$$Q^d_{|X||Y|}(q) = \sum_{|I|=d}\langle\langle f|\mathcal{P}_0(I)\rangle\rangle$$
$$P^{\gamma}_{|Y|}((q_i)_{i\in X}) = \sum_{\theta_I=\gamma}\langle\langle f|\mathcal{P}(I)\rangle\rangle$$
$$Q^{\gamma}_{|Y|}((q_i)_{i\in X}) = \sum_{theta_I=\gamma}\langle\langle f|\mathcal{P}_0(I)\rangle\rangle.$$

Proof. For the last two equations, we use Proposition 2.25 with $m = 1$, and the fact that each function $\omega : X \times Y \to \{0,1\}$ can be written as χ_I with $I \subset I \times Y$, with $\omega_X = \gamma$ equivalent to $\theta_I = \gamma$. We conclude with each $P_1^{\chi_I}$ (resp. $Q_1^{\chi_I}$) replaced by the formula given by Lemma 2.27.

The first two equations are special cases, using $X' = \{*\}$ and $Y' = X \times Y$, and identifying a function $\gamma : X' \to \mathbb{N}$ with its value d. $\square$

2.7 The case $m = 2$

In general, we would like to get a good combinatorial description of the Pfister forms involved in P_m^γ and Q_m^γ, without diagonalizing the m-dimensional forms (unlike in Proposition 2.28), but we do not yet have one to offer. On the other hand, we give a satisfying answer when $m = 2$.

Let $q \in \mathrm{Quad}_2(K)$. Instead of diagonalizing it, let us write it as

$$q = \langle t \rangle \langle\langle \delta \rangle\rangle$$

where $\delta = \det(q) \in K^\times/(K^\times)^2$ is well-defined, and $t \in K^\times$ is only well-defined modulo $G(\langle\langle \delta \rangle\rangle)$. Of course if $q = \langle a, b \rangle$ then $\delta = ab$ and we can take $t = a$ or $t = b$.

Then if $(q_i)_{i \in X} \in (\mathrm{Quad}_2(K))^X$ for some finite X, we can define $\delta_i \in K^\times/(K^\times)^2$ and $t_i \in K^\times/G(\langle\langle \delta_i \rangle\rangle)$ for all $i \in X$. This extends naturally to

$$\begin{array}{cccc} \delta : & \mathcal{P}(X) & \longrightarrow & K^\times/(K^\times)^2 \\ & \{i\} & \longmapsto & -\delta_i \end{array}$$

and for any $Y \subset X$:

$$\begin{array}{cccc} t : & \mathcal{P}(Y) & \longrightarrow & K^\times/G(\langle\langle \delta | \mathcal{P}(Y) \rangle\rangle) \\ & \{i\} & \longmapsto & -t_i \end{array}$$

(note that since t_i is well-defined modulo $G(\langle\langle \delta_i \rangle\rangle)$, for any $I \subset Y$ we get that t_I is well-defined modulo $G(\langle\langle \delta | mathcalP(I) \rangle\rangle)$, and in particular modulo $G(\langle\langle \delta | mathcalP(Y) \rangle\rangle))$.

We see, given the definition, that for any $I \subset X$:

$$\prod_{i \in I} q_i = \langle (-1)^{|I|} t(I) \rangle \langle\langle \delta | \mathcal{P}(I) \rangle\rangle. \tag{8}$$

Definition 2.29. *Let $U, V \subset \mathcal{P}(X)$ be affine subspaces with $\overrightarrow{V} \subset \overrightarrow{U}$. Then we define $\Psi^{V,U}, \Psi_0^{V,U} \in IW((\mathrm{Quad}_2)^X)$ by*

$$\Psi^{V,U}((q_i)_{i \in X}) = \langle\langle t|V; \delta|U \rangle\rangle$$

and

$$\Psi_0^{V,U} = \Psi^{V \cap \mathcal{P}_0(X), U}$$

Furthermore, if $A \subset X$ and $J \subset X \setminus A$, we define $V_{A,J} = J + \mathcal{A}$ and

$$\Psi^{J,A} = \Psi^{V_{J,A}, \mathcal{P}(A \cup J)}, \quad \Psi_0^{J,A} = \Psi_0^{V_{J,A}, \mathcal{P}(A \cup J)}.$$

The fact that those formulas indeed define invariants is clear since the morphisms δ and t are canonically defined from the q_i. The crucial observation is that those invariants take values in general Pfister forms.

Remark 2.30. It might happen that $V \cap \mathcal{P}_0(X) = \emptyset$, in which case $\Psi_0^{V,U} = 0$. Otherwise, the direction of $V \cap \mathcal{P}_0(X)$ is $\overrightarrow{V} \cap \mathcal{P}_0(X)$.

In particular, except when $A = \emptyset$ and $|J|$ is odd, the direction of $V_{J,A} \cap \mathcal{P}_0(X)$ is $\mathcal{P}_0(A)$. When $|J|$ is even, this affine space is actually $J + \mathcal{P}_0(A)$, but when $|J|$ is odd there is no natural basepoint.

Lemma 2.31. *Let $U, V \subset \mathcal{P}(X)$ be affine subspaces with $\overrightarrow{V} \subset \overrightarrow{U}$. Then $\Psi_0^{V,U}$ is the even part of $\Psi^{V,U}$.*

In particular, $\Psi_0^{V,U} \in IW^{\dim(U)+\dim(V)-1}((\mathrm{Quad}_2)^X/\sim)$.

Proof. We see from the definition that if δ' and t' are the morphisms corresponding to the family $(q_i')_{i\in X}$ where $q_i' = \langle\lambda\rangle q_i$, then $\delta' = \delta$, while $t'(I) = t(I)$ if $|I|$ is even and $t'(I) = \langle\lambda\rangle t(I)$ if $|I|$ is odd.

This implies that

$$\begin{aligned}\Psi^{V,U}((\langle\lambda\rangle q_i)_{i\in X}) &= \langle\langle t'|V; \delta'|U\rangle\rangle \\ &= \langle\langle t|V \cap \mathcal{P}_0(X); \delta|U\rangle\rangle + \langle\lambda\rangle\langle\langle t|V \cap \mathcal{P}_1(X); \delta|U\rangle\rangle\end{aligned}$$

which shows that $\Psi_0^{U,V}$ is the even part.

The part about degrees follows simply from Proposition 2.12. □

We now see that those invariants with values in general Pfister forms actually generate all invariants.

Proposition 2.32. *Let $\gamma : X \to \{0,1,2\}$, and let $A = \gamma^{-1}(\{2\}) \subset X$ and $B = \gamma^{-1}(\{2\})$. Then*

$$P_2^\gamma = \sum_{J\subset B} 2^{|B\setminus J|}\Psi^{J,A}Q_2^\gamma \qquad = \sum_{J\subset B} 2^{|B\setminus J|}\Psi_0^{J,A}.$$

Proof. The second formula follows from the first be taking the even part on both sides, so we only prove the first one.

Let $(q_i)_{i\in X} \in (\mathrm{Quad}_2(K))^X$. First observe that from the definition (2) we get

$$\begin{aligned}P_2^0(q_i) &= 1 \\ P_2^1(q_i) &= 2 - q_i \\ P_2^2(q_i) &= \langle\langle\delta_i\rangle\rangle - q_i.\end{aligned}$$

Therefore:

$$\begin{aligned}&P^\gamma((q_i)_{i\in X}) \\ &= \prod_{i\in A}(\langle\langle\delta_i\rangle\rangle - q_i) \times \prod_{i\in B}(2 - q_i) \\ &= \sum_{I\subset A, J\subset B} (-1)^{|I|}\langle(-1)^{|I|}t(I)\rangle\langle\langle\delta|\mathcal{P}(I)\rangle\rangle \cdot \langle\langle\delta|\mathcal{P}(A\setminus I)\rangle\rangle \times (-1)^{|J|}\langle(-1)^{|J|}t(J)\rangle\langle\langle\delta|\mathcal{P}(J)\rangle\rangle \cdot 2^{|B\setminus J|} \\ &= \sum_{I\subset A, J\subset B} 2^{|B\setminus J|}\langle t(I\cup J)\rangle\langle\langle\delta|\mathcal{P}(A\cup J)\rangle\rangle \\ &= \sum_{J\subset B} 2^{|B\setminus J|}\langle\langle t|J + \mathcal{P}(A); \delta|\mathcal{P}(A\cup J)\rangle\rangle.\end{aligned}$$

□

Let us now compare the descriptions of P_2^γ given in Propositions 2.28 and 2.32. So we now allow ourselves to look into each q_i: this gives us $f : \mathcal{P}(X\times Y) \to K^\times/(K^\times)^2$ with some Y with 2 elements, and f encodes a diagonalization of each q_i. Recall that $\pi : X \times Y \to X$ is the projection.

We choose a section $s : X \to X \times Y$ of π. It induces a morphism $s_* : \mathcal{P}(X) \to \mathcal{P}(X \times Y)$. We also define

$$\begin{array}{rccc} \Delta : & \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X \times Y) \\ & I & \longmapsto & I \times Y \end{array}$$

which is $\mathbb{F}_2$-linear.

Note that

$$\mathcal{P}(X \times Y) = s_*(\mathcal{P}(X)) \oplus \Delta(\mathcal{P}(X)) \tag{9}$$

and

$$\mathcal{P}_0(X \times Y) = s_*(\mathcal{P}_0(X)) \oplus \Delta(\mathcal{P}(X)).$$

Lemma 2.33. *For all $U, V \subset \mathcal{P}(X)$ affine subspaces with $\overrightarrow{V} \subset \overrightarrow{U}$, we have*

$$\langle\langle t|V; \delta|U\rangle\rangle = \langle\langle f|s_*V + \Delta(U)\rangle\rangle.$$

Proof. The section s gives us a way to choose representatives in $K^\times/(K^\times)^2$ for each $t(\{i\})$: we take $f(\{s(i)\})$. In practice, if $q_i = \langle a_i, b_i\rangle$ for each I is the diagonalization which defines f, then s picks between a_i or b_i for each i, and those are possible representatives for t_i.

Furthermore, $\delta(\{i\}) = -a_ib_i = f(\{i\} \times Y) = f(\Delta(\{i\}))$. This means that $\delta = f \circ \Delta$ and $t = f \circ s_*$ (at least after taking the classes modulo $G(\langle\langle\delta|U\rangle\rangle)$), which gives the formula. ☐

Therefore when we compare Propositions 2.28 and 2.32, we should have, given $\gamma : X \to \{0, 1, 2\}$, $A = \gamma^{-1}(\{2\}) \subset X$ and $B = \gamma^{-1}(\{2\})$:

$$\sum_{I\subset X\times Y, \theta_I=\gamma} \langle\langle f|\mathcal{P}(I)\rangle\rangle = 2^{|B\setminus J|} \sum_{J\subset B} \langle\langle f|s_*(V_{J,A}) + \Delta(\mathcal{P}(A \cup J))\rangle\rangle \tag{10}$$

$$\sum_{I\subset X\times Y, \theta_I=\gamma} \langle\langle f|\mathcal{P}_0(I)\rangle\rangle = 2^{|B\setminus J|} \sum_{J\subset B} \langle\langle f|s_*(V_{J,A} \cap \mathcal{P}_0(X)) + \Delta(\mathcal{P}(A \cup J))\rangle\rangle. \tag{11}$$

We only prove the first formula, the second one being similar and just necessiting taking extra care about which subsets have even cardinality.

Actually, at this point the result is pure combinatorics and has nothing to do with quadratic forms anymore. Let us consider the group algebra $\mathbb{Z}[\mathcal{P}(X \times Y)]$. For any subset $U \subset \mathcal{P}(X \times Y)$, we set $\sigma(U) = \sum_{x\in U} x \in \mathbb{Z}[\mathcal{P}(X \times Y)]$. Then formula (10) boils down to:

Proposition 2.34. *Let $\gamma : X \to \{0, 1, 2\}$, $A = \gamma^{-1}(\{2\}) \subset X$ and $B = \gamma^{-1}(\{2\})$. Then in $\mathbb{Z}[\mathcal{P}(X \times Y)]$, we have*

$$\sum_{I\subset X\times Y, \theta_I=\gamma} \sigma(\mathcal{P}(I)) = 2^{|B\setminus J|} \sum_{J\subset B} \sigma(s_*(V_{J,A}) + \Delta(\mathcal{P}(A \cup J))).$$

Proof. Given the definition of A, the subsets $I \subset X \times Y$ such that $\theta_I = \gamma$ are exactly those that can be written $I = (A \times Y) \coprod I'$ with $I' \subset B \times Y$ such that π induces a bijection $I' \to B$. We then have $\mathcal{P}(I) = \mathcal{P}(A \times Y) \oplus \mathcal{P}(I')$.

Now given $J \subset B$, we have

$$\begin{aligned} s_*(V_J) + \Delta(\mathcal{P}(A \cup J)) &= s(J) + s_*(\mathcal{P}(A)) + \Delta(\mathcal{P}(A)) + \Delta(\mathcal{P}(J)) \\ &= s(J) + Delta(\mathcal{P}(J)) + \mathcal{P}(A \times Y) \end{aligned}$$

where we used (9).

So we are reduced to

$$\sum_I \sigma(\mathcal{P}(I)) = \sum_{J \subset B} 2^{|B \setminus J|} s(J) \sigma(\Delta(\mathcal{P}(J))) \tag{12}$$

where the left sum is over subsets $I \subset B \times Y$ such that π induces a bijection $I \to B$.

If we fully develop the left sum $\sum_I \sigma(\mathcal{P}(I))$, the elements that appear are the $S \subset B \times Y$ such that π induces an injection $S \to B$, and each such S appears as many times as there are ways to complete S in as I such that π induces a bijection $I \to B$. This means that for each element $i \in B \setminus \pi(S)$ there 2 possible choices for the antecedent of i in I, so in total S appears $2^{B \setminus \pi(S)}$ times.

And actually one can see that $s(J)\sigma(\Delta(\mathcal{P}(J)))$ is exactly the sum of all subsets $S \subset B \times Y$ such that π induces a bijection $S \to J$. Indeed, any such S will correspond in the sum given by σ to the subset $T \subset J$ of elements $i \in J$ such that the antecedent of i in S is not $s(i)$.

This establishes (12). □

3 Invariants of quaternionic forms

3.1 Extending invariants to hermitian forms

Let (A, σ) be a central simple algebra with involution of the first kind over K, and let ε be 1 if σ is orthogonal and -1 if σ is symplectic.

If (B, tau) is another such algebra with involution, with corresponding $\varepsilon' \in \{\pm 1\}$, and if $[A] = [B] \in \mathrm{Br}(K)$, then we can make a choice of hermitian Morita equivalence between (A, σ) and (B, τ), which induces an isomorphism $\mathrm{Herm}_n^{\varepsilon}(A, \sigma) \to \mathrm{Herm}_n^{\varepsilon'}(B, \tau)$. Another choice of equivalence changes this map by a multiplicative scalar constant, and therefore there is a *canonical* morphism $\mathrm{Herm}_n^{\varepsilon}(A, \sigma) \to \mathrm{Herm}_n^{\varepsilon'}(B, \tau)/\sim$ which actually gives a canonical isomorphism $\mathrm{Herm}_n^{\varepsilon}(A, \sigma)(L)/\sim \to \mathrm{Herm}_n^{\varepsilon'}(B, \tau)/\sim$.

In particular, for any splitting extension L/K of A, there is a canonical isomorphism $\mathrm{Herm}_n^{\varepsilon}(A_L, \sigma_L)/\sim \to (\mathrm{Quad}_n)_L/\sim$.

Definition 3.1. *Let $\alpha \in IW(\mathrm{Quad}_n/\sim)$. Then an invariant $\widehat{\alpha} \in IW(\mathrm{Herm}_n^{\varepsilon}(A, \sigma)/\sim)$ is an extension of α if for any splitting extension L/K of A, $\widehat{\alpha}_L$ corresponds to α_L through the canonical isomorphism.*

The same definition applies for cohomological invariants.

Remark 3.2. We see that given the ambiguity of choice in hermitian Morita equivalence, it is not clear what extending non-even invariants would even mean. We could possibly define it as "for every splitting extension, there exists a Morita equivalence such that...", but it is not very satisfying (and there is evidence that this definition does not yield anything interesting anyway).

In [6], we define a commutative $\mathbb{Z}/2\mathbb{Z}$-graded ring

$$\widetilde{GW}^{\varepsilon}(A,\sigma) = GW(K) \oplus GW^{\varepsilon}(A,\sigma)$$

and in [5] we show that it is actually naturally a (graded) pre-λ-ring, with operations $\lambda^d : GW^{\varepsilon}(A,\sigma) \to GW^{\varepsilon}(A,\sigma)$ for d odd, and $\lambda^d : GW^{\varepsilon}(A,\sigma) \to GW(K)$ for d even. Furthermore, any hermitian Morita equivalence between (A,σ) and (B,τ) induces an isomorphism of graded pre-λ-rings $\widetilde{GW}^{\varepsilon}(A,\sigma) \to \widetilde{GW}^{\varepsilon}(B,\tau)$, which restricts to the identity on the neutral component $GW(K)$, and when A is split the operation $\lambda^d : GW^{\varepsilon}(A,\sigma) \to GW(K)$ when d is even corresponds to the usual $\lambda^d : GW(K) \to GW(K)$ through the isomorphism $GW^{\varepsilon}(A,\sigma) \simeq GW(K)$ coming from *any* choice of Morita equivalence (the choice does not matter since when d is even $\lambda^d(\langle a\rangle h) = \lambda^d(h)$ holds for any hermitian form h).

Proposition 3.3. *Any even Witt invariant $\alpha \in IW(\mathrm{Quad}_n / \sim)$ extends to $IW(\mathrm{Herm}_n^{\varepsilon}(A,\sigma)/\sim)$.*

Proof. We know that the λ^d with d even form a basis of $IW(\mathrm{Quad}_n / \sim)$. But from what we explained just above, the λ-operation $\lambda^d : GW^{\varepsilon}(A,\sigma) \to GW(K)$ defines an invariant $IW(\mathrm{Herm}_n^{\varepsilon}(A,\sigma)/\sim)$ which extends $\lambda^d \in IW(\mathrm{Quad}_n / \sim)$. Therefore, all invariants extend. □

Remark 3.4. This method not only shows that all invariants extend, it even gives a somewhat canonical extension: if $\alpha = \sum x_d \lambda^d$ in $IW(\mathrm{Quad}_n)$, then take $\widehat{\alpha} = \sum x_d \lambda^d$ in $IW(\mathrm{Herm}_n^{\varepsilon}(A,\sigma))$. This is uniquely defined (but it does depend on the choice of extending each λ^d by the corresponding operation on hermitian forms, which is definitely a very natural choice, but still a choice).

When we extend an invariant in this manner, we use the same notation α to denote this extension.

Now we ask the question for cohomological invariants: can every even cohomological invariants of quadratic forms be extended to $IC(\mathrm{Herm}_n^{\varepsilon}(A,\sigma))$? We give a positive answer when the index of A is 2.

3.2 Some Pfister forms related to quaternions

Let Q be a quaternion algebra over K, let X be a finite set, and let $(h_i)_{i\in X} \in (\mathrm{Herm}_2^{-1}(Q,\gamma))^X$ be a family of anti-hermitian form over (Q,γ) of reduced dimension 2. We want to extend to such a family the special invariants $\Psi_0^{J,A}$ that we defined in Definition 2.29.

Remember that they depended on morphisms $\delta : \mathcal{P}(X) \to K^\times/(K^\times)^2$ and $t : \mathcal{P}(X) \to K^\times/G(\langle\langle \delta|\mathcal{P}(X)\rangle\rangle)$, which extended the basic relation $q_i = \langle t_i\rangle\langle\langle \delta_i\rangle\rangle$. This relation does not make sense when replacing q_i with a hermitian form h_i, but we can still define δ_i, as it is the discriminant of the form, which is also well-defined for hermitian forms. Precisely, if $h_i = \langle z_i\rangle_\gamma$ for some invertible pure quaternion z_i, then we may take $\delta_i = z_i^2$ (it does not depend on the choice of z_i). So our logical naive extension of $\langle\langle \delta|\mathcal{P}(I)\rangle\rangle$ is:

Definition 3.5. *For any subset $I \subset X$, we define*

$$\pi(I) = \langle\langle z_i^2 \rangle\rangle_{i \in I}.$$

Now since we only want even invariants, we only have to define our equivalent of t on $\mathcal{P}_0(X)$, wich means that we should not try to generalize t_i but rather $t_{\{i,j\}}$, which is characterized by $q_i q_j = \langle t_{\{i,j\}} \rangle \langle\langle \delta_i, \delta_j \rangle\rangle$.

Now the description of $\widetilde{GW}(Q, \gamma)$ in [6] tells us that

$$\langle z_i \rangle_\gamma \cdot \langle z_j \rangle_\gamma = \langle -\operatorname{Trd}_Q(z_i z_j) \rangle \varphi_{z_i, z_j}$$

where φ_{z_i,z_j} is the unique 2-fold Pfister form whose class in $h^2(K)$ is $(z_i^2, z_j^2) + [Q]$. This tells us that our first guess $\pi(I)$ was not the correct one, and also what our $t_{\{i,j\}}$ should be.

Lemma 3.6. *Let $z \in Q_0^\times$. Then $\langle\langle z^2 \rangle\rangle n_Q = 2n_Q$ in $W(K)$.*

Proof. Since $-z^2$ is the reduced norm of z, it is represented by n_Q, and therefore $\langle\langle -z^2 \rangle\rangle n_Q = 0$ in $W(K)$, which means that $\langle z^2 \rangle n_Q = -n_Q$ and thus $\langle\langle z^2 \rangle\rangle n_Q = 2n_Q$ in $W(K)$. □

Proposition 3.7. *If $|I| \geqslant 2$, then $\pi(I) - 2^{|I|-2} n_Q$ is Witt-equivalent to a (unique) $|I|$-fold Pfister form.*

Proof. Assume that $I = \{1, \dots, n\}$. Then from the preceding lemma,

$$2^{n-2} n_Q = \langle\langle z_3^2, \dots, z_n^2 \rangle\rangle n_Q$$

in $W(K)$, which means that

$$\pi(I) - 2^{|I|-2} n_Q = \langle\langle z_3^2, \dots, z_n^2 \rangle\rangle (\langle\langle z_1^2, z_2^2 \rangle\rangle - n_Q).$$

Now if we take $z_0 \in Q_0^\times$ which anti-commutes with z_1:

$$\begin{aligned} \langle\langle z_1^2, z_2^2 \rangle\rangle - n_Q &= \langle\langle z_1^2, z_2^2 \rangle\rangle - \langle\langle z_1^2, z_0^2 \rangle\rangle \\ &= \langle\langle z_1^2, z_0^2 z_2^2 \rangle\rangle - \langle\langle z_1^2, z_0^2, -z_2^2 \rangle\rangle \end{aligned}$$

in $W(K)$, and $\langle\langle z_1^2, z_0^2, -z_2^2 \rangle\rangle$ is hyperbolic since $-z_2^2$ is represented by $n_Q = \langle\langle z_1^2, z_0^2 \rangle\rangle$. So in fact $\langle\langle z_1^2, z_2^2 \rangle\rangle - n_Q$ is Witt-equivalent to a 2-fold Pfister form. We conclude with the fact that two n-fold Pfister forms which are Witt-equivalent are actually isometric. □

This allows the following definition:

Definition 3.8. *For any $I \subset X$, we set $\varphi(I) = \pi(I)$ if $|I| \geqslant 1$, and $\varphi(I)$ is the unique $|I|$-fold Pfister form which is Witt-equivalent to $\pi(I) - 2^{|I|-2} n_Q$.*

Remark 3.9. When Q is split, then n_Q is hyperbolic and therefore $\varphi(I) = \pi(I)$.

Example 3.10. When Q is split, and each h_i corresponds to some q_i though some Morita equivalence, then $\varphi(I) = \Psi^{\emptyset, I}((q_i)_{i \in X})$ (the choice of Morita equivalence does not matter since this is an even invariant). So we can already extend $\Psi^{\emptyset, I}$ to $IW((\mathrm{Herm}_2^{-1}(Q, \gamma))^X)$, to an invariant that also takes values in Pfister forms.

Of course $\pi(I)$ would also define such an extension (but it is not the relevant one for our purposes).

We can record some basic facts about those forms:

Lemma 3.11. *Let $I, J \subset X$. Then*

$$\pi(I)\cdot\pi(J) = 2^{|I\cap J|}\pi(I\cup J)$$
$$\pi(I)\cdot\varphi(J) = \begin{cases} 2^{|I\cap J|}\pi(I\cup J) & \text{if } |J|\geqslant 1 \\ 2^{|I\cap J|}\varphi(I\cup J) & \text{otherwise} \end{cases}$$
$$\varphi(I)\cdot\varphi(J) = \begin{cases} 2^{|I\cap J|}\pi(I\cup J) & \text{if } |I|,|J|\geqslant 1 \\ 2^{|I\cap J|}\varphi(I\cup J) & \text{otherwise.} \end{cases}$$

Proof. The first formula is clear once we notice that $\langle\langle a,a\rangle\rangle = \langle\langle -1,a\rangle\rangle = 2\langle\langle a\rangle\rangle$. The other two are also very simple using Lemma 3.6. □

It is also useful to know more about the case where $|I| = 2$.

Proposition 3.12. *Suppose $I = \{i,j\}$. Then the symmetric bilinear form*

$$\begin{array}{ccc} Q\times Q & \longrightarrow & K \\ (z,z') & \longmapsto & \mathrm{Trd}_Q(zz_i\overline{z'}z_j) \end{array}$$

is isometric to $\langle \mathrm{Trd}_Q(z_iz_j)\rangle\varphi(I)$.

Proof. This is the characterization of $h_j\cdot h_j$ given in [6]. □

3.3 The invariant $\Psi^{J,A}$

We now generalize our morphism t:

Proposition 3.13. *We keep the notations of Section 3.2. There is a unique group morphism*

$$t : \mathcal{P}_0(X) \to K^\times/G(\varphi(X))$$

such that for all $i \neq j \in X$ we have $t(\{i,j\}) = -\mathrm{Trd}_Q(z_iz_j)$.

Proof. The $\{i,j\}$ are generators of $\mathcal{P}_0(X)$ as a group (or equivalently as an $\mathbb{F}_2$-vector space), so there can only be one morphism satisfying those conditions.

The only non-trivial thing to check is that $t(\{i,j\})t(\{i,k\}) = t(\{j,k\})$, in other words that $T = -\mathrm{Trd}_Q(z_iz_j)\,\mathrm{Trd}_Q(z_i,z_k)\,\mathrm{Trd}_Q(z_j,z_k)$ is represented by the Pfister form $\varphi(\{i,j,k\})$.

Using Lemma 3.14, we see that

$$T = -(\mathrm{Trd}_Q(z_jz_k))^2\cdot z_i^2 + (-\mathrm{Trd}_Q(z_jz_k))\cdot\mathrm{Trd}_Q(z_iz_jz_iz_k).$$

Now it follows from Proposition 3.12 (using $z_1 = z_j$, $z_2 = z_k$ and $z = z' = z_i$) that $-\mathrm{Trd}_Q(z_jz_k)\cdot\mathrm{Trd}_Q(z_iz_jz_iz_k)$ is represented by $\varphi(\{j,k\})$. It is also clear that $-(\mathrm{Trd}_Q(z_jz_k))^2\cdot z_i^2$ is represented by $\langle -z_i^2\rangle$, and thus by $\langle -z_i^2\rangle\varphi(\{j,k\}$.

All in all, T is represented by $\langle -z_i^2\rangle\varphi(\{j,k\})\perp\varphi(\{j,k\})$, which is $\langle\langle z_i^2\rangle\rangle\varphi(\{j,k\})$, and therefore $\varphi(\{i,j,k\})$ (see Lemma 3.11). □

The following lemma is used in the proof of Proposition 3.13, but can also be used in other contexts, so we record it here. It is a simple but powerful relation satisfied by quaternions.

Lemma 3.14. *Let $z_1, z_2, z_3 \in Q_0^\times$. Then*

$$\operatorname{Trd}_Q(z_1z_2)\operatorname{Trd}_Q(z_1z_3) = z_1^2\operatorname{Trd}_Q(z_2z_3) + \operatorname{Trd}_Q(z_1z_2z_1z_3).$$

Proof. The basic idea is that if $z, z' \in Q_0^\times$, then $\operatorname{Trd}_Q(zz') = zz' + z'z$. Then:

$$\begin{aligned}
\operatorname{Trd}_Q(z_1z_2)\operatorname{Trd}_Q(z_1z_3) &= \operatorname{Trd}_Q(z_1z_2)(z_1z_3 + z_3z_1) \\
&= \operatorname{Trd}_Q(z_1z_2)z_1z_3 + z_3z_1\operatorname{Trd}_Q(z_1z_2) \\
&= (z_1z_2 + z_2z_1)z_1z_3 + z_3z_1(z_1z_2 + z_2z_1) \\
&= z_1^2(z_2z_3 + z_3z_2) + (z_1z_2z_1z_3 + z_3z_1z_1z_3) \\
&= z_1^2\operatorname{Trd}_Q(z_2z_3) + \operatorname{Trd}_Q(z_1z_2z_1z_3).
\end{aligned}$$ □

Remark 3.15. We get an analog of equation (8): for any $I \in \mathcal{P}_0(X)$,

$$\prod_{i\in I} h_i = \langle t(I)\rangle\varphi(I). \tag{13}$$

Example 3.16. When Q is split and z_i corresponds through some Morita equivalence to $q_i = \langle a_i, b_i\rangle$, then $-\operatorname{Trd}_Q(z_iz_j) = a_ib_j + b_ia_j$. In the split case we had naturally taken as repesentative for $t_{i,j}$ either a_ia_j, a_ib_j, b_ia_j or b_ib_j, but this is another possibility, more symmetrical: indeed, one can check by hand that $a_ia_j(a_ib_j + b_ia_j)$ is represented by $\langle\langle -a_ib_i, -a_jb_j\rangle\rangle$.

Definition 3.17. *For any disjoint subsets $A, J \subset X$, we define $\Psi_0^{J,A} \in IW_0((\mathrm{Herm}_2^{-1}(Q,\gamma))^X)$ by*

$$\Psi_0^{J,A}((h_i)_{i\in X}) = \psi(J;A) = \langle\langle t|W_{J,A}\rangle\rangle\varphi(A\cup J)$$

where $W_{J,A} = (J + \mathcal{P}(A)) \cap \mathcal{P}_0(X)$.

This form is well-defined since $W_{J,A}$ is an affine subset of $\mathcal{P}_0(A\cup J)$. It is clearly an invariant as everything is defined canonically. From everything we said this far we deduce:

Proposition 3.18. *The invariant $\Psi_0^{J,A} \in IW_0((\mathrm{Herm}_2^{-1}(Q,\gamma))^X)$ given in Definition 3.17 is an extension of $\Psi_0^{J,A} \in IW_0((\mathrm{Quad}_2)^X)$ defined in Definition 2.29, which also takes values in general $(2|A| + |J| - 1)$-fold Pfister forms.*

3.4 Extending cohomological invariants to quaternionic forms

Proposition 3.19. *Let $d > 2$ be even, with $d \leqslant 2|X|$. The image of the extended invariant $Q_n^d \in IW(\mathrm{Herm}_{2|X|}^{-1}(Q,\gamma))$ in $IW((\mathrm{Herm}_2^{-1}(Q,\gamma))^X)$ is given by*

$$N(|X|,d)n_Q + \sum_{J,A\subset X}\binom{|X| + |A| - k(J,A)}{|X| + |A| - d}2^{d-k(J,A)}\Psi^{J,A}$$

where $N(|X|,d) \in \mathbb{N}$ is divisible by $2^{d/2-2}$ and $k(J,A) = 2|A| - |J|$, and the sum is over all $J, A \subset X$ disjoint with $2|A| \leqslant d \leqslant |X| + |A|$ and $k(J,A) \leqslant d$.

Proof. Let us write $h = \sum_{i\in X}\langle z_i\rangle_\gamma$. As we mentioned just before equation (3), this equation holds in any pre-λ-ring, and in particular in $\widetilde{GW}(Q,\gamma)$, so

$$P_n^d(h) = \sum_{|\gamma|=d}\prod_{i\in X} P_2^{\gamma(i)}(\langle z_i\rangle_\gamma).$$

Now by definition:

$$\begin{aligned}
&P_2^0(\langle z_i\rangle_\gamma) = 1,\\
&P_2^1(\langle z_i\rangle_\gamma) = 2 - \langle z_i\rangle_\gamma,\\
&P_2^2(\langle z_i\rangle_\gamma) = \langle\langle z_i^2\rangle\rangle - \langle z_i\rangle_\gamma.
\end{aligned}$$

Note that in $\widetilde{GW}(Q,\gamma)$ the $\mathbb{Z}/2\mathbb{Z}$-grading makes it even easier to pick out the even and odd part: $Q_n^d(h)$ is the even component of $P_n^d(h)$, meaning the component in $GW(K)$.

So $Q_2^\gamma(h)$ is the component in $GW(K)$ of

$$P_2^\gamma((\langle z_i\rangle_\gamma)_{i\in X}) = \prod_{i\in A}(\langle\langle z_i^2\rangle\rangle - \langle z_i\rangle_\gamma)\cdot\prod_{i\in B}(2-\langle z_i\rangle_\gamma)$$

where $A = \gamma^{-1}(\{2\})$ and $B = \gamma^{-1}(\{1\})$. When we develop this product, we must choose for each $i \in A\cup B$ whether we choose the term in $GW(K)$ or the term in $GW^{-1}(Q,\gamma)$. We can describe our choice by the subsets $I \subset A$ and $J \subset B$ of the indices for which we chose the odd component. We must have $|I|+|J|$ even, so that the resulting product lands in $GW(K)$. This gives:

$$\begin{aligned}
Q_2^\gamma((\langle z_i\rangle_\gamma)_{i\in X}) &= \sum_{I\cup J\in\mathcal{P}_0(A\cup B)} 2^{|B\setminus J|}\pi(A\setminus I)\langle t(I\cup J)\rangle\varphi(I\cup J)\\
&= 2^{|B|}\pi(A) + \sum_{I\cup J\in\mathcal{P}_0(A\cup B)\setminus\{\emptyset\}} 2^{|B\setminus J|}\langle t(I\cup J)\rangle\varphi(A\cup J)\\
&= 2^{|A\cup B|-2}n_Q + \sum_{I\cup J\in\mathcal{P}_0(A\cup B)} 2^{|B\setminus J|}\langle t(I\cup J)\rangle\varphi(A\cup J)\\
&= 2^{|A\cup B|-2}n_Q + \sum_{J\subset B} 2^{|B\setminus J|}\left(\sum_{I\subset A, |I\cup J| \text{ even}}\langle t(I\cup J)\rangle\right)\varphi(A\cup J)\\
&= 2^{|A\cup B|-2}n_Q + \sum_{J\subset B} 2^{|B\setminus J|}\psi_0(J;A).
\end{aligned}$$

Now we just have to count how many times n_Q and each $\psi(J;A)$ appears when we sum over all $\gamma : X\to\mathbb{N}$ with $|\gamma| = d$. Note that $|\gamma| = 2|A|+|B|$. Of course the data of γ is equivalent to the data of A and B. So as for n_Q, we see that it appears

$$\sum_{A\subset X}\binom{|X|-|A|}{d-2|A|}2^{d-|A|-2}$$

times, where the sum is over subsets A such that $2|A| \leqslant d \leqslant |X|+|A|$. We call that number $N(|X|,d)$, and we see that all 2-powers appearing in the sum are with exponent $d-|A|-2 \geqslant d/2-2$.

For each pair (J,A) of disjoint subsets of X, $\psi^{J,A}$ appears with factor $2^{|B\setminus J|} = 2^{d-k(J,A)}$ for each overset B of J which is disjoint with A and with $2|A|+|B| = d$. For this to exist we must have $2|A| \leqslant d \leqslant |X|+|A|$ and $k(J,A) \leqslant d$, and in that case there are $\binom{|X|+|A|-k(J,A)}{|X|+|A|-d}$ possibilities for B (by choosing $B \setminus J$ in $X \setminus (A \cup J)$ with cardinal $d-k(J,A)$). □

Theorem 3.20. *Every even invariant $\alpha \in IC_0(\mathrm{Quad}_{2n})$ extends to an invariant $\widehat{\alpha} \in IC(\mathrm{Herm}^{\varepsilon}_{2n}(A,\sigma)/\sim)$.*

Proof. We may assume that $\alpha \in IC_0^m(\mathrm{Quad}_{2n})$ According to Proposition 2.23, α is liftable to some $\beta \in IW_0^{\geqslant m}(\mathrm{Quad}_{2n})$. Then write $\beta = \sum_d a_d Q_n^d$ with $a_d \in I^{m-d+1}(K)$. We can then extend naturally β to hermitian forms, and then set

$$\widehat{beta} = \beta - \sum_d a_d N(n,d) n_Q$$

which according to Proposition 3.19 takes values in I^m (indeed, it is a combination of invariants $a_d 2^{d-k(J,A)}\Psi_0^{J,A}$ and $\Psi_0^{J,A}$ takes values in general $(k(J,A)-1)$-fold Pfister forms). Since $n_Q = 0 \in W(K)$ when Q is split, the class of $\widehat{\beta}$ in $IC_0(\mathrm{Herm}^{\varepsilon}_{2n}(A,\sigma))$ extends α. □

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