Lambda-operations for hermitian forms over algebras with involution of the first kind

Nicolas Garrel

Introduction

In his seminal work [5], Serre defines a general notion of *invariants* of a certain class of objects over a base field: if K is a field, and A and H are two functors defined on the category of field extensions of K to the category of sets, then the set $\mathrm{Inv}(A, H)$ of invariants of A with values in H is defined as the set of natural transformations from A to H. Often, H is actually a functor to the category of rings, and $\mathrm{Inv}(A, H)$ is then naturally an $H(K)$-algebra. The idea is very simple and broad: for each extention L/K, to each element of $A(L)$ we associate an element of $H(L)$, with the only constraint that this is compatible with field extensions.

Some primary sources of examples are given by $A(L) = H^1(L, G(L))$ (non-abelian cohomology) where G is an algebraic group over K, and $H(L) = H^*(L, \mathbb{Z}/2\mathbb{Z})$ (mod 2 Galois cohomology) or $H(L) = W(L)$ (the Witt group of quadratic forms). This is the study of so-called (mod 2) cohomological invariants and Witt invariants of algebraic groups, which is a very active field of study. When G is the orthogonal group of some non-degenerate quadratic form of rank n, the corresponding $A(L)$ can be identified with the set $\mathrm{Quad}_n(L)$ of isometry classes of non-degenerate quadratic forms of rank n, thus we are looking at invariants of quadratic forms of fixed dimension.

In this special case, Serre gives a complete description of the invariants: the cohomological invariants of Quad_n are a free $H^*(K, \mathbb{Z}/2\mathbb{Z})$-module with basis the Stiefel-Whitney invariants $w_0, \dots, w_n$ [5, 17.3], and the Witt invariants are a free $W(K)$-module over the so-called λ-powers $\lambda^0, \dots, \lambda^n$ [5, 27.16]. Those operations, introduced by Bourbaki in [2], can be described either explicitly given a diagonalization:

$$\lambda^d(\langle a_1, \dots, a_n\rangle) = \sum_I \langle a_I\rangle$$

where I runs over the subsets of $\{1, \dots, n\}$ of cardinal d and $a_I = \prod_{i\in I} a_i$; or it can be described more intrinsically given a bilinear space (V, b):

$$\lambda^d(b) : \begin{array}{ccc} \Lambda^d(V) \times \Lambda^d(V) & \longrightarrow & K \\ (u_1 \wedge \cdots \wedge u_d, v_1 \wedge \cdots \wedge v_d) & \longmapsto & \det\left(b(u_i, v_j)\right). \end{array}$$

Those operations, though very natural, make suprisingly few appearances in the quadratic form literature; for instance, they do not even get a passing mention in references such as [11], [3], [13] or [16]. This might be in part due to the fact that they are not well-defined on the Witt ring, which is traditionally the preferred algebraic structure for working with quadratic forms, but rather on the Grothendieck-Witt ring $GW(K)$. The appearance of the Grothendieck-Witt ring in this context should not come as a surprise: the theory of λ-rings was initiated by Grothendieck to be applied to K-theory, where it had most of its success, and $GW(K)$ is nothing but the 0th hermitian K-theory ring of K. As an illustration of the relative indifference given to the λ-operations on quadratic forms, the fact that $GW(K)$ forms a λ-ring has only been proved somewhat recently [12], although the corresponding (and technically more difficult) fact for the K_0 ring of a variaty was shown in the realy days of K-theory.

It seems that, in the quest for cohomological invariants, the proof of the Milnor conjecture by Voevodsky has spurred some resurgence of interest in those operations through the following strategy: since we now have access to canonical morphisms $e_n : I^n(K) \to H^n(K, \mathbb{Z}/2\mathbb{Z})$ (where $I^n(K)$ is the nth power of the fundamental ideal $I(K)$ of the Witt ring), we can define cohomological invariants by constructing invariants with values in I^n. We will say that a mod 2 cohomological invariant is *liftable* if it is obtainable this way. As a basic example, it is not difficult to show, using λ-operations, that the Stiefel-Whitney invariants are liftable. In [7], we strongly rely on λ-operations to describe all cohomological invariants of I^n, which turn out to be all liftable. This was inspired by previous constructions by Rost [15] (improved by Garibaldi in [4]), already using λ-operations to define some liftable invariants of spin groups.

When the algebraic group G is a classical group, we are led to consider invariants of central simple algebras with involution. To implement our strategy, we need to be able to associate quadratic forms to those objects. The most common such construction is given by trace forms: if (A, σ) is an algebra with involution of the first kind, we can define the trace form $T_A : x \mapsto \mathrm{Trd}_A(x^2)$, the involution trace form $T_\sigma : x \mapsto \mathrm{Trd}_A(x\sigma(x))$, its restriction T_σ^+ to the subspace of σ-symmetric elements, and its restriction T_σ^- to the subspace of anti-symmetric elements. These forms are related by $T_A = T_\sigma^+ - T_\sigma^-$ and $T_\sigma = T_\sigma^+ + T_\sigma^-$, so it is enough to know T_σ^+ and T_σ^-. They have indeed been used to define or compute some cohomological invariants, for instance in [1] or [14].

If (A, σ) is a central simple algebra with involution of the first kind over K, and h is an ε-hermitian form over (A, σ) (with $\varepsilon = \pm 1$), we define in this article the λ-power $\lambda^d(h)$ for any $d \in \mathbb{N}$, which is a quadratic form over K if d is even, and an ε-hermitian form over (A, σ) if d is odd. In fact, we defined in [7] a graded commutative ring

$$\widetilde{GW}(A, \sigma) = GW(K) \oplus GW^{-1}(K) \oplus GW(A, \sigma) \oplus GW^{-1}(A, \sigma),$$

and we define here a graded pre-λ-ring structure on this ring.

This gives a whole new means of associating a quadratic form to an ε-hermtian form, using $h \mapsto \sum_d a_d \lambda^{2d}(h)$ for some $a_d \in W(K)$, and also to an algebra with involution (A, σ), applying this method to the canonical diagonal form $h = \langle 1 \rangle_\sigma$. We recover as a special case the trace forms mentioned above, using $\lambda^2(\langle 1 \rangle_\sigma)$ (see Corollary (3.30)).

The grading on the ring $\widetilde{GW}(A, \sigma)$ that we are interested in is a crucial part of the theory, and we are therefore led to develop a theory of graded pre-λ-rings. Actually, $\widetilde{GW}(A, \sigma)$ is defined as a quotient of a graded ring $\widehat{GW}(A, \sigma)$, which is graded not over a group as is more commonplace, but over a monoid. In addition, it is itself the Grothendieck ring of a graded semiring $\widehat{SW}(A,\sigma)$. It is thus on $\widehat{SW}(A,\sigma)$ that our constructions are initially carried out, in order to be carried over to $\widetilde{GW}(A,\sigma)$ (in two steps). This means that we should in fact develop a theory of graded pre-λ-semirings, which are graded over a monoid. The first section of the article is dedicated to the layout of this theory; as we do not expect that the reader is familiar in any way with λ-rings, and as rings graded over a monoid are a somewhat esoteric subject, this section is meant to be rather didactic.

In Section 2, we briefly recall (without proofs) the necessary constructions and results regarding $\widetilde{GW}(A,\sigma)$ from [7]. Then Section 3 is the heart of the article, where the structure of graded pre-λ-ring is established on $\widetilde{GW}(A,\sigma)$. The remaining Section 4 is devoted to the determinant, and the duality it induces on λ-powers.

Preliminaries and conventions

We fix throughout the article a base field K of characteristic not 2. An arbitrary field extension of K will usually be denoted by L, and if X is any object (algebra, module, quadratic form, etc.) defined over K, then $X_L = X \otimes_K L$ is the corresponding object over L, obtained by base change. All algebras and modules are assumed to be finite-dimensional over K.

Commutative monoids

Let M be a commutative monoid. We write $M^\times$ for the subgroup of invertible elements (even when M is denoted additively). We say that a submonoid $N \subset M$ is saturated if whenever $x+y=z$ with $x,z \in N$ then $y \in N$; if M is actually a group, this exactly means that N is a subgroup.

We write $G(M)$ for the Grothendieck group of M, which we recall is generated by formal differences of elements of M; there is always a monoid morphism $M \to G(M)$ but it is only injective if M satisfies the cancellation property.

Bilinear forms

We systematically identify symmetric bilinear forms and quadratic forms over K, through $b \mapsto q_b$ with $q_b(x) = b(x,x)$. Diagonal quadratic forms are denoted $\langle a_1, \dots, a_n\rangle$, with $a_i \in K^*$, and the n-fold Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ is $\langle 1, -a_1\rangle \cdots \langle 1, -a_n\rangle$. Every bilinear form is implicitly assumed to be nondegenerate.

For $\varepsilon = \pm 1$, we use $SW^\varepsilon(K)$ (as in "semi Witt") to designate the monoid of isometry classes of (nondegenerate) ε-symmetric bilinear forms over K; we do include the zero-dimensional form. Then the Grothendieck-Witt group $GW^\varepsilon(K)$ is the Grothendieck group of $SW^\varepsilon(K)$, and the Witt group $W^\varepsilon(K)$ is the quotient of $GW^\varepsilon(K)$ by the hyperbolic forms. We often omit the superscript ε when $\varepsilon = 1$. Recall that $SW(K)$ is a commutative semiring (for the orthogonal sum and tensor product operations), and $GW(K)$ and $W(K)$ are commutative rings, as well as $GW^\pm(K) = GW(K) \oplus GW^{-1}(K)$ and $W^\pm(K) = W(K) \oplus W^{-1}(K)$.

Central simple algebras

If A is a central simple algebra over K, we write $\mathrm{Trd}_A : A \to K$ for the reduced trace of A, and $\mathrm{Nrd}_A : A \to K$ for its reduced norm (recall that they can be defined as descents of the usual trace and determinant maps on endormorphism algebras of vector spaces). The trace form of A is the symmetric bilinear form $T_A : A \times A \to K$ defined by $T_A(x, y) = \mathrm{Trd}_A(xy)$.

If V is a right A-module, its reduced dimension $\mathrm{rdim}(V)$ is characterized by $\deg(A)\,\mathrm{rdim}(A) = \dim_K(A)$; if V is non-zero, it is the degree of the central simple algebra $\mathrm{End}_A(V)$.

We denote by F_A (resp. $F_{A,2}$) the generic splitting field of A (resp. the field of generic reduction to index 2), defined as the function field of the Severi-Brauer variety $SB(A)$ of A (resp. the generalized Severi-Brauer variety $SB_2(A)$, see [10, 1.C]).

Algebras with involution

When we say that (A, σ) is an algebra with involution over K, we mean that A is a central simple algebra over K, and that σ is an involution of the first kind on A, so σ is an anti-automorphism of K-algebra of A, with $\sigma^2 = \mathrm{Id}_A$. In general, "involution" will be synonym with "involution of the first kind".

For $\varepsilon = \pm 1$, we define the set $\mathrm{Sym}^\varepsilon(A, \sigma) \subset A$ of ε-symmetric elements of (A, σ), meaning that they satisfy $\sigma(a) = \varepsilon a$, and $\mathrm{Sym}^\varepsilon(A^\times, \sigma)$ is the subset of invertible ε-symmetric elements. If A has degree n, the involution σ is orthogonal when $\mathrm{Sym}(A, \sigma)$ has dimension $n(n + 1)/2$, and it is symplectic when it has dimension $n(n - 1)/2$. In particular, (K, Id) is an algebra with orthogonal involution. A quaternion algebra admits a unique symplectic involution, called its canonical involution, and we denote it by γ. We define the sign $\varepsilon(\sigma)$ of an involution σ as 1 if σ is orthogonal, and -1 if it is symplectic.

The involution trace form of (A, σ) is the bilinear form $T_\sigma : A \times A \to K$ defined by $T_\sigma(x, y) = \mathrm{Trd}_A(\sigma(x)y)$. The subspaces $\mathrm{Sym}^\varepsilon(A, \sigma) \subset A$ for $\varepsilon = \pm 1$ are orthogonal for T_σ, and we write T_σ^ε for its restriction to $\mathrm{Sym}^\varepsilon(A, \sigma) \subset A$.

Hermitian forms

If (A, σ) is an algebra with involution, and $\varepsilon = \pm 1$, an ε-hermitian module (V, h) over (A, σ) is a *right* A-module V, together with an ε-hermitian form $h : V \times V \to A$ (always assumed to be non-degenerate). We often just speak about an ε-hermitian form without mentioning the underlying module. If we want to talk about an ε-hermitian form without mentioning ε explicitly, we will use the expression "general hermitian form".

If V is non-zero, h induces the so-called adjoint involution σ_h on the central simple algebra $\mathrm{End}_A(V)$, characterized by $h(u(x), y) = h(x, \sigma_h(u)(y))$. We call $\varepsilon(h) = \varepsilon$ the sign of h, and $t(h) = \varepsilon(\sigma_h)$ the type of h; notice that $t(h) = \varepsilon(\sigma)\varepsilon(h)$.

If $a \in \mathrm{Sym}^\varepsilon(A^\times, \sigma)$, the elementary diagonal ε-hermitian form $\langle a\rangle_\sigma : A\times A \to A$ is defined by $(x, y) \mapsto \sigma(x)ay$. A diagonal form $\langle a_1, \ldots, a_n\rangle_\sigma$ is then the (orthgonal) sum of the $\langle a_i\rangle_\sigma$.

We define the monoid $SW^\varepsilon(A, \sigma)$ of isometry classes of ε-hermitian forms over (A, σ) (the zero module is included), and $SW_\varepsilon(A, \sigma)$ is the monoid of forms of *type* ε. Note that $SW_\varepsilon(A, \sigma)$ is equal to $SW^{\varepsilon\varepsilon(\sigma)}(A, \sigma)$. We then define the groups $GW^\varepsilon(A,\sigma)$, $GW_\varepsilon(A,\sigma)$, $W^\varepsilon(A,\sigma)$ and $W_\varepsilon(A,\sigma)$ as we did for bilinear forms (which are the special case where $(A,\sigma)=(K,\mathrm{Id})$).

Let us record the two following facts, which are direct consequences of the main results in [8]:

Lemma 0.1. *Let (A,σ) be an algebra with involution. Then the scalar extension map $W_1(A,\sigma)\to W_1(A_{F_A},\sigma_{F_A})$ is injective.*

Lemma 0.2. *Let (A,σ) be an algebra with involution. Then the scalar extension map $W_{-1}(A,\sigma)\to W_{-1}(A_{F_{A,2}},\sigma_{F_{A,2}})$ is injective.*

Symmetric functions

We define the ring of symmetric functions $U_\mathbb{Z}\subset\mathbb{Z}[[x_1,x_2,\dots]]$ as the subring consisting of the formal series that are invariant under all permutations of the x_i and that are bounded in total degree. For any $f\in U_\mathbb{Z}$ and any $n\in\mathbb{N}^*$, the polynomial $f(x_1,\dots,x_n,0,0,\dots)$ is symmetric in n variables, so for any elements $a_1,\dots,a_n$ in a commutative ring, we can define $f(a_1,\dots,a_n)$ which is symmetric in the a_n. For any $d\in\mathbb{N}$, there is the elementary symmetric function $\sigma_d\in U_\mathbb{Z}$ defined as $\sigma_d=\sum_{i_1<\dots<i_d}x_{i_1}\cdots x_{i_d}$, and the fundamental theorem of symmetric functions states that $U_\mathbb{Z}\simeq\mathbb{Z}[\sigma_1,\sigma_2,\dots]$.

1 Graded pre-λ-semirings

The goal of this article is to define and study an appropriate structure of graded pre-λ-ring on the mixed Grothendieck-Witt ring $\widetilde{GW}(A,\sigma)$. But ultimately this comes from a similar structure on $\widehat{SW}(A,\sigma)$, which is only a semiring, graded over a monoid. Therefore this is the framework that we develop in this section.

We do not assume that the reader is familier with λ-rings or with rings graded over monoids, and we try to give a self-contained account of what is needed for the article. We take [17] as our main reference for the classical theory of (ungraded) λ-rings (though we also sometimes refer to [18]). We make all the necessary adjustments to take the gradings into account (working with semirings instead of rings poses no problem whatsoever), and refer directly to the proofs in [17] when they are completely straightforward to adapt to our context.

1.1 Motivation for λ-rings

The motivation for the notion of a λ-ring is to study the formal properties of exterior powers of (projective) modules or vector bundles (in their various flavours). A common aspect of those objects is that they satisfy a "splitting principle": they can be "split" into sums of elementary "line elements" (for a finitely generated projective module, this corresponds to their characterization as being locally free, for instance in the étale topology). Thus one can regard an element x as being "secretly" a sum $x=l_1+\cdots+l_n$ of "1-dimensional" elements, and we can ask how much of the l_i we can recover from x.

Clearly, only symmetric expressions in the l_i can be extracted. So for any symmetric function $f\in U_\mathbb{Z}$, we would like to have some operation $\widehat{f}$ such that $\widehat{f}(x)=f(l_1,\dots,l_n)$. Of course, from the fundamental theorem of symmetric functions it is enough to consider the case $f = \sigma_d$; the corresponding operation is what will be denoted λ^d.

Thus a λ-ring is a commutative ring endowed with some operations $\lambda^d : R \to R$ which must satisfy certain formal properties that guarantee that they behave as the elementary symmetric functions. In this section we only consider pre-λ-rings, which satisfy a subset of those properties, and we discuss the full notion of λ-ring in section **??**. Note that for some authors, they are respectively called "λ-rings" and "special λ-rings"; we follow what seems to be the current terminology.

1.2 Graded semirings

If M is a commutative monoid (which we usually denote additively), an M-graded commutative monoid A is a commutative monoid endowed with a decomposition $A = \bigoplus_{g\in M} A_g$. Any ungraded commutative monoid can be seen as a trivially graded monoid, meaning it is (uniquely) graded over the trivial monoid. The elements of each A_g are called homogeneous, and the set of homogeneous elements is denoted $|A|$. The degree map $\partial : |A| \to M \cup \{\infty\}$ sends $a \in A_g \setminus \{0\}$ to its degree $g \in M$, and $\partial(0) = \infty$ (where ∞ is a formal element). A subset of A is said to be homogeneous if it contains the homogeneous components (ie the component in each A_g) of all its elements.

If A and B are M-graded, then a graded morphism $f : A \to B$ is a group morphism such that $f(A_g) \subset B_g$ for all $g \in M$. Given a monoid morphism $\varphi : M \to N$, there is an induced N-grading $\varphi_*(A)$ on A, where for each $h \in N$, $A_h = \bigoplus_{\varphi(g)=h} A_g$ (for instance, if $\varphi : M \to \{0\}$ is the trivial morphism, then $\varphi_*(A)$ is just A seen as a trivially graded ring). If A is M-graded and B is N-graded, a lax graded morphism $A \to B$ is the data of some $\varphi : M \to N$, and a graded morphism $f : \varphi_*(A) \to B$; we also say that f is a φ-graded morphism.

Recall that a semiring is the same as a ring except that its underlying additive structure is only that of a commutative monoid, not necessarily a group. An M-graded semiring is a semiring R which is M-graded as an additive monoid, such that $1 \in R_0$ (the neutral component), and $R_g \cdot R_h \subset R_{g+h}$ for any $g, h \in M$. All graded semirings in this article will be commutative. A (lax) graded semiring morphism is a (lax) graded morphism which is also a semiring morphism. Note that any ungraded semiring is naturally a graded semiring for the trivial grading.

The subset $|R| \subset R$ is actually a multiplicative submonoid, and $\partial : |R| \to M \cup \{\infty\}$ is a monoid morphism (where $m + \infty = \infty$ for all $m \in M$). An element $x \in |R|$ is called graded-invertible if for any $g \in M$, multiplication by x induces an additive isomorphism from R_g to $R_{g+\partial(x)}$. The set of graded-invertible elements is denoted by $R^\times$ (which agrees with the usual notation if R is ungraded), and it is a saturated submonoid of $|R|$. A homogeneous element $x \in |R|$ is invertible if and only if it is graded-invertible and $\partial(x)$ is invertible in M; in particular, if M is a group, then $R^\times = |R|^\times$ (the group of invertible elements of the monoid $|R|$). On the other hand, in general an invertible element or R does not have to be graded-invertible if it is not homogeneous.

If R is an M-graded commutative semiring, then the monoid semiring $R[N]$ is a commutative $(M \times N)$-graded semiring. In particular, if R is ungraded, $R[M]$ is M-graded. Note that $|R[N]| \simeq |R| \times N$ as monoids, and $R^\times \times N^\times$ is a submonoid of $(R[N])^\times$.

An augmentation on a commutative M-graded semiring R is a graded morphism $\widetilde{\delta}_R : R \to \mathbb{Z}[M]$, and a morphism of augmented graded semirings is a graded morphism which preserves the augmentation. We also define the total augmentation $\delta_R : R \to \mathbb{Z}$, which is an ungraded semiring morphism, as the composition of $\widetilde{\delta}_R$ and the canonical ring morphism $\mathbb{Z}[M] \to \mathbb{Z}$.

1.3 Pre-λ-semirings

Definition 1.1. *Let M be a commutative monoid. An M-graded pre-λ-semiring is an M-graded commutative semiring R endowed with functions $\lambda^d : R \to R$ for all $d \in \mathbb{N}$ such that:*

- *for all $g \in M$ and $d \in \mathbb{N}$, $\lambda^d(R_g) \subset R_{dg}$;*
- *for all $x \in R$, $\lambda^0(x) = 1$ and $\lambda^1(x) = x$;*
- *for all $g \in M$, $x, y \in R_g$ and $d \in \mathbb{N}$, $\lambda^d(x + y) = \sum_{p+q=d} \lambda^p(x)\lambda^q(y)$.*

Note that by definition it is enough to define the λ^d on each homogeneous component; therebis then a unique extension to the whole semiring satisfying the axioms.

Example 1.2. If (V, b) is an ε-symmetric bilinear space over K, then there is a natural ε^d-symmetric bilinear map $\lambda^d(b)$ on $\Lambda^d(V)$, given by

$$\lambda^d(b)(u_1 \wedge \cdots \wedge u_d, v_1 \wedge \cdots \wedge v_d) = \det(b(u_i, v_j)).$$

This defines a $\mu_2(K)$-graded pre-λ-semiring structure on $SW^{\pm}(K)$.

Of course, a morphism of graded pre-λ-semirings, which we call a graded λ-morphism, is a graded semiring morphism which commutes with the operations λ^d. This defines a category of M-graded pre-λ-semirings.

For any symmetric function $f \in U_{\mathbb{Z}}$, we define an operation $\widehat{f} : R \to R$: if $f = P(\sigma_1, \dots, \sigma_d)$ for some (uniquely defined) $P \in \mathbb{Z}[x_1, \dots, x_d]$, then $\widehat{f}(x) = P(\lambda^1(x), \dots, \lambda^d(x))$. Then by definition $\widehat{\sigma_d} = \lambda^d$, and any $\widehat{f}$ commutes with any λ-morphism. Also note that if f is homogeneous of degree n, then $\widehat{f}(R_g) \subset R_{ng}$ for all $g \in M$.

Definition 1.3. *Let R be an M-graded pre-λ-semiring, and let $x \in R$. We define the λ-dimension $\dim_\lambda(x)$ of x as the supremum in $\mathbb{N} \cup \{\infty\}$ of all $n \in \mathbb{N}$ such that $\lambda^n(x) \neq 0$. The subset of elements with finite λ-dimension is denoted $R^{f.d.}$.*

We usually just say "dimension" for the λ-dimension when there is no risk of confusion. Note that only $0 \in R$ has dimension 0, that $\dim_\lambda(x + y) \leqslant \dim_\lambda(x) + \dim_\lambda(y)$, and that if f is a graded λ-morphism, $\dim_\lambda(f(x)) \leqslant \dim_\lambda(x)$.

Then if $x = l_1 + \cdots + l_n$ where the l_i have dimension 1, and if we assume that any product of the l_i still has dimension (at most) 1, we can easily show that for any $f \in U_{\mathbb{Z}}$, $\widehat{f}(x) = f(l_1, \dots, l_n) \in R$. We will see that for the cases we consider all products of elements of dimension 1 have dimension 1, so we can indeed extract all symmetric functions of the l_i from x, as we discussed ealier (of course such a sum decomposition does not exist in general).

It can be useful to rephrase the definition of a graded pre-λ-semiring in a more abstract way. For any commutative M-graded semiring R, consider the commutative multiplicative monoid

$$\Lambda(R) = 1 + tR[[t]] \subset R[[t]] \tag{1}$$

where $R[[t]]$ is of course the semiring of formal series over R, and define for any $g \in M$ the submonoid

$$\Lambda(R)_g = \left\{ \sum a_d t^d \in \Lambda(R) \,|\, \forall n \in \mathbb{N},\, a_d \in R_{dg} \right\}. \tag{2}$$

Then we get an M-graded monoid

$$\Lambda_M(R) = \bigoplus_{g \in M} \Lambda(R)_g. \tag{3}$$

There is a natural graded monoid morphism $\eta_R : \Lambda_M(R) \to R$ (where R is seen as an additive monoid) which sends a series $\sum a_d t^d$ to a_1. Defining functions $\lambda^d : R_g \to R_{dg}$ for all $g \in M$ and $d \in \mathbb{N}^*$ is the same as defining a single homogeneous function $\lambda_t : R \to \Lambda_M(R)$, using $\lambda_t(x) = 1 + \sum_{d>0} \lambda^d(x) t^d$, and from the definition of the monoid structure on $\Lambda(R)$ one can easily check that the λ^d define a graded pre-λ-semiring structure if and only if λ_t is an additive morphism which is a section of η_R.

If $f : S \to R$ is any M-graded semiring morphism, then it induces a commutative diagram

$$\begin{array}{ccc} \Lambda_M(S) & \xrightarrow{\eta_S} & S \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f} \\ \Lambda_M(R) & \xrightarrow{\eta_R} & R \end{array}$$

and when R and S are graded pre-λ-semirings, then f is a λ-morphism if and only if the following natural diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\lambda_t} & \Lambda_M(S) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \\ R & \xrightarrow{\lambda_t} & \Lambda_M(R). \end{array}$$

An element $x \in R$ is finite-dimensional exactly when $\lambda_t(x) \in R[[t]]$ is a polynomial, and its dimension is then the degree of this polynomial.

Remark 1.4. If $\varphi : M \to N$ is a monoid morphism, there is a canonical morphism $\Lambda_M(R) \to \Lambda_N(\varphi_*(R))$, which is compatible with the construction of η_R. This means that a structure of M-graded pre-λ-semiring on R canonically induces a structure of N-graded pre-λ-semiring on $\varphi_*(R)$. In particular, taking φ to be the trivial morphism $\varphi : M \to \{0\}$, it induces a structure of (ungraded) pre-λ-semiring on R (as $\Lambda_{\{0\}}(\varphi_*(R))$ is just $\Lambda(R)$).

1.4 Augmentation

The following proposition is immediate from the pre-λ-semiring axioms:

Proposition 1.5. *If R is an M-graded pre-λ-semiring and N is a commutative monoid, then $R[N]$ has a canonical $(M \times N)$-graded pre-λ-semiring structure given by $\lambda^d(x \cdot h) = \lambda^d(x) \cdot (dh)$ for all $d \in \mathbb{N}$, $x \in R$ and $h \in N$.*

Moreover, the canonical semiring morphism $R[H] \to R$ is a λ-morphism.

Example 1.6. There is a canonical pre-λ-ring structure on $\mathbb{Z}$, given by $\lambda^d(n) = \binom{n}{d}$, which then induces a canonical M-graded pre-λ-ring structure on the monoid ring $\mathbb{Z}[M]$.

Definition 1.7. *An augmentation of an M-graded pre-λ-semiring R is an augmentation $\widetilde{\delta_R} : R \to \mathbb{Z}[M]$ which is a λ-morphism. The total augmentation map $\delta_R : R \to \mathbb{Z}$ is then an ungraded λ-morphism.*

Example 1.8. The graded dimension map $SW^{\pm}(K) \to \mathbb{Z}[\mu_2]$, which sends the isometry class of (V, b) in $SW^{\varepsilon}(K)$ to $\dim(V) \cdot \varepsilon$, is an augmentation on the graded pre-λ-semiring $SW^{\pm}(K)$.

In general, we want to give the augmentation map the interpretation of a "graded dimension", but it should not be confused with the λ-dimension. Since for a strictly negative n, $\binom{n}{d} \neq 0$ for all $d \in \mathbb{N}$, an element $x \in R^{f.d.}$ must satisfy $\delta_R(x) \geqslant 0$. Also, since the λ-dimension of $n \in \mathbb{N}$ is just n, and a λ-morphism lowers the λ-dimension, we see that we must have $0 \leqslant \delta_R(x) \leqslant \dim_\lambda(x)$.

1.5 Positive structure

In practice, a lot of (pre)-λ-rings, such as the K_0 ring of a commutative ring or a topological space, are defined as Grothendieck rings of semirings (of modules or vector bundles in those examples), which means that general elements are formal differences of "concrete" elements which enjoy a better behaviour.

Proposition 1.9. *Let S be an (augmented) M-graded pre-λ-semiring. There is a unique structure of (augmented) M-graded pre-λ-ring on the Grothendieck ring $G(S)$ such that the canonical morphism $S \to G(S)$ preserves the structure.*

Furthermore, if $S \mapsto R$ is a morphism of (augmented) M-graded pre-λ-semirings where R is actually a ring, then the induced ring morphism $G(S) \to R$ is a morphism of (augmented) M-graded pre-λ-rings.

Proof. The fact that $G(S)$ has a unique compatible ring structure is classical (and easy), and clearly $G(S)_g = G(S_g)$ for all $g \in M$ is the unique compatible grading. Interpreting the λ-structure as a monoid morphism $\lambda_t : S \to \Lambda_M(S)$, and observing that $\Lambda_M(G(S))$ is actually a group since $G(S)$ is a ring, the universal property of Grothendieck groups tells us that there is a unique $\lambda_t : G(S) \to \Lambda_M(G(S))$ such that the natural diagram

$$\begin{array}{ccc} S & \xrightarrow{\lambda_t} & \Lambda_M(S) \\ \downarrow & & \downarrow \\ G(S) & \xrightarrow{\lambda_t} & \Lambda_M(G(S)) \end{array}$$

commutes.

If R is an M-graded pre-λ-ring and $S \to R$ is a λ-morphism, then we need to check whether the diagram of abelian groups

$$\begin{array}{ccc} G(S) & \xrightarrow{\lambda_t} & \Lambda_M(G(S)) \\ \downarrow & & \downarrow \\ R & \xrightarrow{\lambda_t} & \Lambda_M(R) \end{array}$$

commutes. But since both compositions $G(S) \to \Lambda_M(R)$ extend $S \to \Lambda_M(R)$, this is true bu unicersal property.

The case where S is augmented is proved similarly, as the augmentation is just the data of a λ-morphism $S \to \mathbb{Z}[M]$. □

Remark 1.10. If R is an augmented M-graded pre-λ-ring and $S \subset R$ is a substructure which is only a semiring and generates R additively, then $R \simeq G(S)$ canonically as an augmented M-graded pre-λ-ring.

Example 1.11. We have $GW^{\pm}(K) \simeq G(SW^{\pm}(K))$ as augmented $\mu_2(K)$-graded pre-λ-rings.

Lemma 1.12. *Let S be an M-graded semiring. Then the canonical map $f : S \to G(S)$ satisfies $f(S^\times) \subset R^\times$.*

Proof. If $a \in S^\times$, then multiplication by a induces isomorphisms $S_g \to S_{g+\partial(a)}$ for all $g \in M$, and therefore multiplication by $f(a)$ induces isomorphisms $G(S_g) \to G(S_{g+\partial(a)})$. □

As we mentioned, in that situation we would like the sub-semiring $S \subset R$ to enjoy good properties, that will somewhat extend to R. We adapt the treatment in [18] to formalize those properties:

Definition 1.13. *Let S be an augmented M-graded pre-λ-semiring. We say that S is* rigid *if*

1. *for any $x \in |S|$, $\delta_S(x) = \dim_\lambda(x)$;*
2. *if $\ell(S)$ is the set of 1-dimensional homogeneous elements, called* line elements, *then $\ell(S) \subset S^\times$.*

Example 1.14. Our usual example $SW^{\pm}(K)$ is rigid. The line elements are the 1-dimensional quadratic forms in $SW(K)$.

Definition 1.15. *Let R be an augmented M-graded pre-λ-semiring. We say that R is an M-structured semiring if it is either a rigid semiring, or is a ring and has a distinguished rigid sub-semiring $S \subset R$ (which is part of the data) which generates R additively. In that last case S is called a* positive structure *on R.*

In both cases the given rigid sub-semiring is denoted $R_{\geqslant 0}$, its elements are called positive, *and we also set $R_{>0} = R_{\geqslant 0} \setminus \{0\}$. We also write $\ell(R) = \ell(R_{\geqslant 0})$.*

Note that there is an obvious category of M-structured rings, which preserve the positive structure.

Remark 1.16. The two cases in the definition of M-structured semirings are very different: indeed, in a rigid semiring, the only element which has an additive inverse is 0.

Example 1.17. If R is an M-structured semiring, then $R[N]$ is an $(M \times N)$-structured semiring, with $(R[N])_{\geqslant 0} = (R_{\geqslant 0})[N]$ and $\ell(R[N]) = \ell(R) \times N$.

As $\mathbb{N} \subset \mathbb{Z}$ is a positive structure for $\mathbb{Z}$, $\mathbb{N}[M]$ is a positive structure for $\mathbb{Z}[M]$ with $\ell(\mathbb{Z}[M]) = M$.

Lemma 1.18. *Let R be an M-structured semiring. Then $\ell(R)$ is a saturated submonoid of $R^\times$, and therefore a saturated submonoid of the multiplicative monoid $|R|$.*

Proof. If R is rigid, then $\ell(R) \subset R^\times$ by definition, and if R is a ring then it follows from Lemma 1.12. Since $R^\times$ is saturated in $|R|$, then it is enough to show that $\ell(R)$ is saturated in $R^\times$.

Let $x, y, z \in R^\times$ such that $xy = z$ and $x, z \in \ell(R)$. If R is rigid, then the equality $\delta_R(x)\delta_R(y) = \delta(z)$ gives $1 \times \delta_R(y) = 1$ so $\dim_\lambda(y) = 1$ and by definition y is a line element.

In particular, when R is a ring we only need to prove that y is positive. Since x is graded-invertible in $R_{\geqslant 0}$ and $\partial(z) = \partial(x)+\partial(y)$, we may write $z = x \cdot y'$ with y' positive of degree $\partial(y)$, and since x is graded-invertible in R, then $y = y'$. $\square$

Note that positive elements have finite dimension since they are sums of homogeneous positive elements. The positive structure ensures that R enjoys a well-behaved theory of dimension:

Proposition 1.19. *Let R be an M-structured semiring. Then for any element $x \in R^{f.d.}$, we have $\dim_\lambda(x) = \delta_R(x)$, and the leading coefficient of $\lambda_t(x)$ is a line element. In particular, all elements of λ-dimension 1 are line elements, and $\dim_\lambda$ is an additive function on $R^{f.d.}$.*

Proof. Let $A \subset R$ be the subset of elements x such that $\lambda_t(x)$ is a polynomial of degree $\delta_R(x)$ whose leading coefficient is a line element. By definition $A \subset R^{f.d}$, and we want to show that they are actually equal, which takes care of all statements in the proposition.

First, we see that $|R_{\geqslant 0}| \subset A$. Indeed, if x is a positive homogeneous element, then $\dim_\lambda(x) = \delta_R(x)$ by hypothesis, and if this dimension is n, then $\lambda^n(x)$ is a line element because it is positive and has dimension $\binom{n}{n} = 1$.

Then, we see that A is stable by sum: if $x, y \in A$, then $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$, so if at^n and bt^m are the leading terms of $\lambda_t(x)$ and $\lambda_t(y)$ respectively, the leading term of $\lambda_t(x+y)$ is abt^{n+m} because $ab \in \ell(R)$ is non-zero. Note that $n+m$ is $\delta_R(x) + \delta_R(y) = \delta_R(x+y)$.

This shows that $R_{\geqslant 0} \subset A$; if R is rigid we are done, otherwise R is a ring and we show that if $x, y \in A$ are such that $x - y$ has finite dimension, then $x - y \in A$. We may assume that x and y are homogeneous. Let at^n, bt^m and ct^r be the leading terms of $\lambda_t(x)$, $\lambda_t(y)$, and $\lambda_t(x-y)$ respectively; we have $a, b \in \ell(R)$ and $r = \dim_\lambda(x-y) \geqslant \delta_R(x-y) = m - n$. But since $\lambda_t(y)\lambda_t(x-y) = \lambda_t(x)$, we must actually have $r = m - n$ and $ac = b$. Since $\ell(R)$ is saturated in $|R|$, $c \in \ell(R)$. $\square$

Remark 1.20. A morphism of M-structured rings preserves the augmentation, therefore it preserves the dimension of finite-dimensional elements, and also induces a monoid morphism between line elements.

1.6 Duality

A lot of λ-rings also come with a notion of duality, such as the K_0 of a commutative ring, with the dual of a finite projective module. As far as we know, this subsection and the next are original material even for ungraded pre-λ-rings.

Definition 1.21. *Let R be an M-structured semiring. A duality on R is the data of:*

- *an involutive monoid endomorphism $\Delta_M : g \mapsto g^*$ of M;*
- *an involutive Δ_M-lax graded λ-endomorphism $\Delta_R : x \mapsto x^*$ of R;*
- *a monoid morphism $\omega : M \to \ell(R)$;*

such that

- *for all $g \in M$, $\omega(g^*) = \omega(g)^*$;*
- *for all $g \in M$, $\partial(\omega(g)) = g + g^*$;*
- *for all $x \in \ell(R)$, $\omega(\partial(x)) = x \cdot x^*$.*

The $\omega(g)$ are called the dualizing elements of R.

Note that given a duality on R, for any $g, h \in M$, multiplication by $\omega(h)$ gives a canonical isomorphism between R_g and R_{g+h+h^*}.

Remark 1.22. When M and $\ell(R)$ are 2-torsion group, then there is a canonical duality where Δ_M and Δ_R are the identity, and ω is the trivial morphism. In that case we say that R is self-dual.

Remark 1.23. If R is an M-structured ring, any duality on $R_{\geqslant 0}$ exends uniquely to a duality on R, and if $R_{\geqslant 0}$ is self-dual then so is R.

Example 1.24. It is easy to see that $SW(K)^{\pm}$ and $GW^{\pm}(K)$ are self-dual.

Example 1.25. If R is an M-structured semiring with duality, and $h \mapsto h^*$ is any involutive endomorphism $N \to N$ (for instance the identity), then $R[N]$ has a natural duality extending the one on R, using $\Delta_{M\times N}(g, h) = (g^*, h^*)$, $\Delta_{R[N]}(x \cdot h) = x^* \cdot h^*$, and $\omega(g, h) = \omega(g) \cdot (h + h^*)$.

1.7 Determinant

We saw in proposition 1.19 that if R is M-structured and $x \in R$ has dimension n, then $\lambda^n(x)$ is a line element. This construction can be extended to all elements when R is a ring (in the rigid case no extension is necessary).

Proposition 1.26. *Let R be an M-structured ring. There is a unique group morphism*

$$\det : R \to G(\ell(R)),$$

which we call the determinant*, such that if* $\dim_\lambda(x) = n \in \mathbb{N}$*, then* $\det(x) = \lambda^n(x)$*. If f is a morphism of M-structured rings, then* $\det(f(x)) = f(\det(x))$.

Proof. We saw in proposition 1.19 that this was well-defined for finite-dimensional elements, and in particular for positive elements. If $x \in R$ is any element, we may write $x = x_1 - x_2$ where the x_i are positive, and we set $\det(x) = \det(x_1)\det(x_2)^{-1}$. This is well-defined since if $x = y_1 - y_2$ is another decomposition, then $x_1 + y_2 = x_2 + y_1$ so $\det(x_1)\det(y_2) = \det(x_2)\det(y_1)$. The fact that this defines a group morphism is then clear by definition.

The compatibility with morphisms is easy to see since they preserve the λ-dimension (see Remark 1.20) and $f(\lambda^n(x)) = \lambda^n(f(x))$ if x is of finite dimension n. □

Remark 1.27. Recall that if M is actually a group, then $R^\times$ is a group, as well as $\ell(R)$ (as it is a saturated submonoid), so in that case $G(\ell(R)) = \ell(R)$.

Example 1.28. The determinant of an element of $GW^\pm(K)$ is a 1-dimensional quadratic form, which is basically the same thing as a square class, and it is easy to check that this square class is indeed the usual determinant of a bilinear form (defined as the square class of the determinant of its matrix in any basis). In particular, it is just $\langle 1\rangle$ for all elements in $GW^{-1}(K)$.

The determinant often interacts with duality:

Definition 1.29. *Let R be an M-structured semiring with duality. We say that R has determinant duality if for any homogeneous positive element $x \in R$ of finite dimension $n \in \mathbb{N}$, and any $p, q \in \mathbb{N}$ such that $p + q = n$, we have*

$$\omega(q\partial(x))\lambda^p(x) = \det(x)\lambda^q(x^*).$$

Remark 1.30. Obviously, when R is a ring, saying that R or $R_{\geqslant 0}$ has determinant duality is the same.

Example 1.31. It is not very hard to see that $SW^\pm(K)$ (and therefore $GW^\pm(K)$) has determinant duality. Indeed, for elements of $SW^{-1}(K)$ it is trivial because they are classified by their dimension, so the formula amounts to $\binom{n}{p} = \binom{n}{q}$. For quadratic forms, this may be checked on a diagonalization. Indeed, if $q = \langle a_1, \dots, a_n\rangle$ then $\lambda^d(q)$ has a diagonalization given by the a_I where $I \subset \{1, \dots, n\}$ has size d, and $a_I = \prod_{i\in I} a_i$, so $\lambda^n(x)\langle a_I\rangle = \langle a_{\overline{I}}\rangle$ shows the formula (where $\overline{I}$ is the complementary of I).

2 Mixed Grothendieck-Witt rings

In this section, we review the definitions and results from [6] about the mixed Grothendieck-Witt ring which are necessary for our purposes (we refer to [6] for the proofs and details).

Definition 2.1. *Let (A, σ) and (B, τ) be algebras with involution over K. A hermitian Morita equivalence from (B, τ) to (A, σ) is a B-A-bimodule V endowed with a regular ε-hermitian form $h : V \times V \to A$ over (A, σ) (with $\varepsilon = \pm 1$), such that the action of B on V induces a K-algebra isomorphism $B \simeq \mathrm{End}_A(V)$, under which τ is sent to the adjoint involution σ_h (which means that $h(bu, v) = h(u, \tau(b)v)$).*

There exists such an equivalence if and only if A and B are Brauer-equivalent; in this case, the isomorphism class of the bimodule V is unique, and if we fix such a V, the ε-hermitian form h is unique up to a multiplicative scalar: if h' is another choice, there is some $\lambda \in K^\times$ such that $h' = \langle\lambda\rangle h$.

Definition 2.2. *The hermitian Brauer 2-group $\mathbf{Br}_h(K)$ of K is the category whose objects are algebras with involutions over K, and morphisms $(B,\tau) \to (A,\sigma)$ are isomorphism classes of ε-hermitian Morita equivalences from (B,τ) to (A,σ).*

The composition of $(U,g) : (C,\theta) \to (B,\tau)$ and $(V,h) : (B,\tau) \to (A,\sigma)$ is defined as $(U \otimes_B V, f)$ with

$$f(u \otimes v, u' \otimes v') = h(v, g(u,u')v').$$

Note that the identity of (A,σ) in $\mathbf{Br}_h(K)$ is the diagonal form $(A, \langle 1\rangle_\sigma)$. It can be shown that all morphisms are invertible. Specifically, if (V,h) is a morphism from (B,τ) to (A,σ), then we can define an A-B-bimodule $\overline{V}$ as being V as a K-vector space, but with twisted action $a \cdot v \cdot b = \tau(b) \cdot v \cdot \sigma(a)$. Then we have a natural form $\overline{h}$ on $\overline{V}$ over (B,τ) defined by $\overline{h}(x,y)z = xh(y,z)$ for all $x,y,z \in V$, and the inverse of (V,h) in $\mathbf{Br}_h(K)$ is $(\overline{V}, \langle\varepsilon(h)\rangle\overline{h})$.

If $\varepsilon = \pm 1$ then SW_ε is a functor from $\mathbf{Br}_h(K)$ to commutative monoids; this fact is the main reason that SW_ε is preferred to the more usual SW^ε. The category $\mathbf{Br}_h(K)$ is monoidal for the usual tensor product of algebras and (hermitian) modules over K, and $SW_1 \oplus SW_{-1}$ is a monoidal functor from $\mathbf{Br}_h(K)$ to $\mu_2(K)$-graded commutative monoids; this means that there is a natural map

$$SW_\varepsilon(A,\sigma) \otimes_K SW_{\varepsilon'}(B,\tau) \to SW_{\varepsilon\varepsilon'}(A \otimes_K B, \sigma \otimes \tau)$$

which is just given by the tensor product of hermitian modules.

For any (A,σ), this means that if $\Gamma' = \mathbb{N} \times \mu_2(K)$ we can define a Γ'-graded semiring $\widehat{SW}(A,\sigma)$ by

$$\widehat{SW}(A,\sigma)_{(d,\varepsilon)} = SW_\varepsilon(A^{\otimes d}, \sigma^{\otimes d})$$

and that $\widehat{SW}$ is a functor from $\mathbf{Br}_h(K)$ to Γ'-graded semirings. This also define the associated Grothendieck ring $\widehat{GW}(A,\sigma)$, with

$$\widehat{GW}(A,\sigma)_{(d,\varepsilon)} = GW_\varepsilon(A^{\otimes d}, \sigma^{\otimes d}).$$

We want to use the fact that algebras with involution have exponent 2 to reduce the $\mathbb{Z}$-grading to a $\mathbb{Z}/2\mathbb{Z}$-grading.

Definition 2.3. *Let (A,σ) be an algebra with involution over K. The mixed Grothendieck-Witt monoid of (A,σ) is the Gamma-graded commutative monoid $\widetilde{SW}(A,\sigma)$, where $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mu_2(K)$, defined by*

$$\widetilde{SW}(A,\sigma)_{(0,\varepsilon)} = SW^\varepsilon(K) \text{ and } \widetilde{SW}(A,\sigma)_{(1,\varepsilon)} = SW_\varepsilon(A,\sigma).$$

The corresponding Grothendieck group $\widetilde{GW}(A,\sigma)$ is the mixed Grothendieck-Witt group of (A,σ).

Then $\widetilde{SW}$ is a functor from $\mathbf{Br}_h(K)$ to the category of Γ-graded commutative monoids, the components $SW(K)$ and $SW^{-1}(K)$ being constant along this functor. The components $SW(K)$ and $SW^{-1}(K)$ are called *even*, and the $SW_\varepsilon(A,\sigma)$ are *odd* (this is in agreement with the $\mathbb{Z}/2\mathbb{Z}$-grading induced by the Γ-grading). We also call $SW_1(A,\sigma)$ the *orthogonal* component, and $SW_{-1}(A,\sigma)$ the *symplectic* one. All of this also applies to $\widetilde{GW}(A,\sigma)$.

Sending a module to its reduced dimension defines a monoid morphism from $SW_\varepsilon(A,\sigma)$ to $\mathbb{N}\subset\mathbb{Z}$. They can be bundled together to define a Γ-graded monoid morphism $\widetilde{\mathrm{rdim}} : \widetilde{SW}(A,\sigma)\to\mathbb{Z}[\Gamma]$ which we call the graded reduced dimension. The total reduced dimension $\mathrm{rdim} : \widetilde{SW}(A,\sigma)\to\mathbb{Z}$ is the composition of $\widetilde{\mathrm{rdim}}$ with the canonical morphism $\mathbb{Z}[\Gamma]\to\mathbb{Z}$. The same thing can of course also be done to define a Γ'-graded group morphism $\widehat{\mathrm{rdim}}$ from $\widehat{SW}(A,\sigma)$ to $\mathbb{Z}[\Gamma']$.

The following theorem is the key to the semiring structure on $\widetilde{SW}(A,\sigma)$. Let us write $|A|_\sigma$ for the left $(A\otimes_K A)$-module which is A as a vector space, with "twisted" action

$$(a\otimes b)\cdot x = ax\sigma(b). \tag{4}$$

It can then be checked that $(|A|_\sigma, T_\sigma)$ is a morphism in $\mathbf{Br}_h(K)$ from $(A^{\otimes 2},\sigma^{\otimes 2})$ to (K,Id).

Theorem 2.4. *Let (A,σ) be an algebra with involution over K, and let $d,r\in\mathbb{N}$ be integers of the same parity. Then there is a canonical hermitian Morita equivalence*

$$\varphi^{d,r}_{(A,\sigma)} : (A^{\otimes d},\sigma^{\otimes d})\to(A^{\otimes r},\sigma^{\otimes r})$$

such that:

- $\varphi^{d,d}_{(A,\sigma)}$ *is the identity of* $(A^{\otimes d},\sigma^{\otimes d})$*;*
- $\varphi^{2,0}_{(A,\sigma)}$ *is* $(|A|_\sigma, T_\sigma)$*;*
- $\varphi^{r_1+r_2,s}_{(A,\sigma)}\circ(\varphi^{d_1,r_1}_{(A,\sigma)}\otimes\varphi^{d_1,r_1}_{(A,\sigma)}) = \varphi^{d_1+d_2,s}_{(A,\sigma)}$*;*
- *if $f : (B,\tau)\to(A,\sigma)$ is any morphism in $\mathbf{Br}_h(K)$, then $\varphi^{d,r}_{(A,\sigma)}\circ f^{\otimes d} = f^{\otimes r}\circ\varphi^{d,r}_{(B,\tau)}$.*

Remark 2.5. The theorem can be rephrased as saying that there is a canonical monoidal functor from the discrete monoidal category $\mathbb{Z}/2\mathbb{Z}$ to $\mathbf{Br}_h(K)$ sending 1 to (A,σ), and that this contruction is itself functorial in (A,σ).

There is an obvious morphism $\Gamma'\to\Gamma$ given by $\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$, and using $\varphi^{d,r}_{(A,\sigma)}$ with $r\in\{0,1\}$, we can define a canonical lax graded monoid morphism

$$\widehat{SW}(A,\sigma)\longrightarrow\widetilde{SW}(A,\sigma).$$

Note that it is by definition the identity on the components $SW_\varepsilon(K)$ and $SW_\varepsilon(A,\sigma)$, and in general it is an isomorphism on each homogeneous component (but it is not a graded isomorphism since $\Gamma'\to\Gamma$ is not an isomorphism).

Theorem 2.6. *Let (A,σ) be an algebra with involution over K. The Γ'-graded semiring $\widehat{SW}(A,\sigma)$ is commutative, and $\widehat{\mathrm{rdim}}$ is a graded augmentation.*

There is a unique Γ-graded semiring structure on $\widetilde{SW}(A,\sigma)$ such that the canonical map $\widehat{SW}(A,\sigma) \to \widetilde{SW}(A,\sigma)$ is a semiring morphism. Then $\widetilde{SW}(A,\sigma)$ is a commutative Γ-graded semiring, $\widetilde{\mathrm{rdim}}$ is a graded augmentation, and the functor $\widetilde{SW}$ is a functor from $\mathbf{Br}_h(K)$ to the category of augmented Γ-graded commutative semirings.

Example 2.7. By definition, in $\widetilde{SW}(A,\sigma)$ we have $\langle 1\rangle_\sigma^2 = T_\sigma \in SW(K)$.

Remark 2.8. The functoriality implies that the graded semiring $\widetilde{SW}(A,\sigma)$ only depends on the Brauer class of A, but *noncanonically*: if A and B are Brauer-equivalent, then there exists a Morita equivalence between (A,σ) and (B,τ) inducing an isomorphism on the mixed Grothendieck-Witt semirings, but there are several choices of such equivalences, which amount to a choice of scaling, which only has influence on the odd components.

This contruction is of course compatible with taking Grothendieck groups, and also with taking the group quotient $W_\varepsilon(A^{\otimes d},\sigma^{\otimes d})$ of $GW_\varepsilon(A^{\otimes d},\sigma^{\otimes d})$. All in all, this defines a natural commutative diagram of lax graded semiring morphisms, which is functorial over $\mathbf{Br}_h(K)$:

$$\begin{array}{ccccc}
\widehat{SW}(A,\sigma) & \longrightarrow & \widehat{GW}(A,\sigma) & \longrightarrow & \widehat{W}(A,\sigma) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{SW}(A,\sigma) & \longrightarrow & \widetilde{GW}(A,\sigma) & \longrightarrow & \widetilde{W}(A,\sigma).
\end{array}$$

Example 2.9. If $(A,\sigma) = (K,\mathrm{Id})$, then $\widetilde{SW}(K,\mathrm{Id})$ is canonically isomorphic to the group semiring $SW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$. The Γ-grading keeps track of the even/odd components. Of course, $\widetilde{GW}(K,\mathrm{Id}) \simeq GW^{\pm}(K)[\mathbb{Z}/2\mathbb{Z}]$ also.

Example 2.10. If (Q,γ) is a quaternion algebra over K with its canonical involution, then for any $a,b\in K^\times$ we have in $\widetilde{GW}(Q,\gamma)$:

$$\langle a\rangle_\gamma \cdot \langle b\rangle_\gamma = \langle 2ab\rangle n_Q \tag{5}$$

where n_Q is the norm form of Q, and for any invertible pure quaternions z_1 and z_2:

$$\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma = \langle -\mathrm{Trd}_Q(z_1z_2)\rangle \varphi_{z_1,z_2} \tag{6}$$

where φ_{z_1,z_2} is the unique 2-fold Pfister form whose Clifford invariant $e_2(\varphi_{z_1,z_2})$ is the Brauer class $[Q] + (z_1^2, z_2^2)$. Note that $\mathrm{Trd}_Q(z_1z_2) = 0$ exactly when z_1 and z_2 anti-commute, and in that case φ_{z_1,z_2} is hyperbolic; the formula above should then be understood as saying that $\langle z_1\rangle_\gamma \cdot \langle z_2\rangle_\gamma$ is hyperbolic.

3 λ-operations on hermitian forms

3.1 Alternating powers of a module

The first step is to associate to each A-module V an $A^{\otimes d}$-module $\mathrm{Alt}^d(V)$, such that we recover the construction of the exterior power in the split case. The natural context of the exterior power construction for vector spaces is that of Schur functors, but the development of such a theory for modules over central simple algebras is beyond the scope of this article, and will be addressed in future work.

It is still useful to view the exterior power construction as a consequence of the structure of module over a symmetric group. Namely, if V is a K-vector space, then $V^{\otimes d}$ is naturally a left $K[\mathfrak{S}_d]$-module, and $\Lambda^d(V)$ is the quotient of $V^{\otimes d}$ by the subspace generated by the kernels of $1-\tau$ for all transpositions $\tau \in \mathfrak{S}_d$. Now if V is a right A-module, it is in particular a K-vector space, so $V^{\otimes d}$ still has the left $K[\mathfrak{S}_d]$-module structure given by the permutation of the d factors, but it is *not* the one we want to use, since it is not compatible with the action of $A^{\otimes d}$ on the right (to see how ill-suited this action would be, consider that if $V = A$, the $A^{\otimes d}$-module generated by the kernel of any $1-\tau$ is the full $V^{\otimes d}$, since it contains $1_A \otimes \cdots \otimes 1_A$).

Instead, recall from [10, 3.5] that for any central simple algebra B, the Goldman element $g_B \in (B \otimes_K B)^\times$ is defined as the pre-image of the reduced trace map $\mathrm{Trd}_B : B \to K \subset B$ under the canonical isomorphism of *vector spaces*

$$B \otimes_K B \xrightarrow{\sim} B \otimes_K B^{op} \xrightarrow{\sim} \mathrm{End}_K(B),$$

and from [10, 10.1] that sending a transposition $(i,\, i+1) \in \mathfrak{S}_d$ to $1 \otimes \cdots \otimes g_B \otimes \cdots \otimes 1$ extends to a group morphism $\mathfrak{S}_d \to (B^{\otimes d})^\times$, and thus to a K-algebra morphism

$$K[\mathfrak{S}_d] \to B^{\otimes d}.$$

Now let again V be a (non-zero) right A-module, and let $B = \mathrm{End}_A(V)$. Then from the canonical algebra morphisms from $K[\mathfrak{S}_d]$ to $B^{\otimes d}$ and $A^{\otimes d}$, we have a canonical structure of left $K[\mathfrak{S}_d]$-module on $V^{\otimes d}$ which commutes with the action of $A^{\otimes d}$, and a canonical structure of right $K[\mathfrak{S}_d]$-module which commutes with the action of $B^{\otimes d}$ (in particular, those two actions commute with one another). Those two actions are by default the ones we have in mind when we work with $V^{\otimes d}$. When $V = 0$, $\mathrm{End}_A(V)$ is not a central simple algebra, but we of course still have (trivial) actions of $K[\mathfrak{S}_d]$. Note that both actions are compatible with scalar extension, and the one on the left is compatible with Morita equivalence. The connection with the permutation action is given by:

Proposition 3.1. *Let A be a central simple algebra over K, let V be a right A-module, and set $B = \mathrm{End}_A(V)$. Then for any $v_1, \dots, v_d \in V$ and any $\pi \in \mathfrak{S}_d$:*

$$\pi(v_1 \otimes \cdots \otimes v_d)\pi^{-1} = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)}.$$

If $A = K$, the action of $K[\mathfrak{S}_d]$ on $V^{\otimes d}$ on the right is trivial, and its action on the left is the usual permutation action on the d factors.

Proof. By construction of the $K[\mathfrak{S}_d]$-module structures, we can reduce to the case where $d = 2$ and π is the transposition, and extending the scalars if necessary we may assume that A and B are split.

In this case we have $A \simeq \mathrm{End}_K(U)$, $B \simeq \mathrm{End}_K(W)$, and $V \simeq \mathrm{Hom}_K(U, W)$ with obvious actions from A and B. It is shown in [10] that the Goldman element g_A in $A \otimes_K A \simeq \mathrm{End}_K(U \otimes_K U)$ is the permutation map (and of course likewise for g_B). Therefore, if $f_1, f_2 \in V$ and $u_1, u_2 \in U$:

$$\begin{aligned}(g_B \cdot f_1 \otimes f_2)(u_1 \otimes u_2) &= g_B(f_1(u_1) \otimes f_2(u_2)) \\ &= f_2(u_2) \otimes f_1(u_1) \\ &= (f_2 \otimes f_1)(g_A(u_1 \otimes u_2)) \\ &= (f_2 \otimes f_1 \cdot g_A)(u_1 \otimes u_2)\end{aligned}$$

so indeed $g_B \cdot f_1 \otimes f_2 = f_2 \otimes f_1 \cdot g_A$.

The last statement is a direct consequence, taking into account that the Goldman element of K is $1 \in K \otimes K = K$. □

For any finite set X, we define the anti-symmetrizer element $s_X \in K[\mathfrak{S}_X]$ by:

$$s_X = \sum_{\pi \in \mathfrak{S}_X} (-1)^\pi \pi. \tag{7}$$

In particular, this defines $s_d \in K[\mathfrak{S}_d]$.

Lemma 3.2. *Let V be a right A-module, where A is a central simple algebra. Let us identify elements of $K[\mathfrak{S}_d]$ with the maps they induce on $V^{\otimes d}$. Then*

$$\ker(s_d) = \sum_{g \neq 1} \ker(1 + (-1)^g g)$$

and

$$\mathrm{Im}(s_d) = \bigcap_g \ker(1 - (-1)^g g).$$

Moreover, the equalities still hold if we restrict g to a generating set.

Proof. In both cases, the equality can be checked after extending the scalars to a splitting field, and then by Morita equivalence it can be reduced to $A = K$, that is to the case of vector spaces, where it amounts to simple combinatorics on a basis, which we spell out explicitly.

Choose a basis $(e_i)_{1 \leqslant i \leqslant r}$ of V. For any $\overline{x} \in \{1, \dots, r\}^d$, we write $e_{\overline{x}}$ for the corresponding basis element of $V^{\otimes d}$, and for any $I \subset \{1, \dots, r\}$ of size d, we define e_I as $e_{\overline{I}}$ where $\overline{I}$ consists of the elements of I in increasing order.

It is easy to see that the kernel of s_d is generated by the $e_{\overline{x}}$ where $\overline{x}$ has at least two equal components (in which case it is in the kernel of $1 - g$ for some transposition g), and by the $e_{\overline{x}} - (-1)^g e_{g\overline{x}}$ where $\overline{x}$ has distinct components and $g \neq 1$, which is in the image of $1 - (-1)^g g$, so in the kernel of $1 - (-1)^g g$. This shows that $\ker(s_d) \subseteq \sum_g \ker(1 + (-1)^g g)$, and the reverse inclusion is obvious since s_d is a multiple of $1 + (-1)^g g$. Note that if $g = g_1 \cdots g_r$ where the g_i are in some generating set, then

$$e_{\overline{x}} - (-1)^g e_{g\overline{x}} = (e_{\overline{x}} + e_{g_r\overline{x}}) - (e_{g_r\overline{x}} + e_{g_{r-1}g_r\overline{x}}) + \cdots - (-1)^g (e_{g_2 \cdots g_r \overline{x}} + e_{g\overline{x}})$$

so we can indeed restrict to g being in that generating set.

It is also easy to see that the $s_d e_I$ form a basis of the image of s_d. Let $v = \sum a_{\overline{x}} e_{\overline{x}} \in \ker(1 - (-1)^g g)$, and let $\overline{x} \in \{1, \dots, r\}^d$. If $\overline{x}$ has at least two equal components then $a_{\overline{x}} = 0$ because there is a transposition g such that $g\overline{x} = \overline{x}$ so $a_{\overline{x}} = -a_{\overline{x}}$ (and we assumed that the characteristic of K is not 2). And if $\overline{x}$ has distinct components, we have $\overline{x} = g\overline{I}$ for some subset I, and $a_{\overline{x}} = (-1)^g a_{\overline{I}}$. All in all, this means that $v = \sum_I a_{\overline{I}} s_d e_I$, so $\bigcap_g \ker(1-(-1)^g g) \subseteq \mathrm{Im}(s_d)$. The reverse inclusion is clear since $(1-(-1)^g g)s_d = 0$. For the last statement of the lemma, note that if we have $gv = (-1)^g v$ when g is in some generating set, then it is true for any g. □

Note that the second equality really uses the fact that the characteristic is not 2, while the first is valid in any characteristic (this is why we added the condition $g \neq 1$ in the sum, which is only needed in characteristic 2).

This lemma shows that, given the classical definition of exterior powers, when V is a vector space (so $A = K$) we have a canonical identification $\mathrm{Alt}^d(V) \simeq \Lambda^d(V)$, with $s_d(v_1 \otimes \cdots \otimes v_d)$ corresponding to $v_1 \wedge \cdots \wedge v_d$. This motivates the following definition:

Definition 3.3. *Let A be a central simple algebra over K, let V be a right A-module, and let $d \in \mathbb{N}$. We set*

$$\mathrm{Alt}^d(V) = s_d V^{\otimes d} \subset V^{\otimes d}$$

as a right $A^{\otimes d}$-module, with in particular $\mathrm{Alt}^0(V) = K$ and $\mathrm{Alt}^1(V) = V$.

Proposition 3.4. *Let A be a central simple algebra over K, let V be a right A-module, and let $d \in \mathbb{N}$. Then*

$$\mathrm{rdim}_{A^{\otimes d}}(\mathrm{Alt}^d(V)) = \binom{\mathrm{rdim}_A(V)}{d}.$$

In particular, if $d > \mathrm{rdim}_A(V)$ then $\mathrm{Alt}^d(V)$ is the zero module.

Proof. Once again, it is enough to check this when A is split, and then by Morita equivalence when $A = K$. But then this is the usual formula for the dimension of $\Lambda^d(V)$. □

Remark 3.5. In [10, §10.A], the algebra $\lambda^d(A)$ is defined, using our notation, as $\mathrm{End}_{A^{\otimes d}}(\mathrm{Alt}^d(A))$, where A is seen as a module over itself. When $d \leqslant \deg(A)$, $\lambda^d(A)$ is a central simple algebra, but when $d > \deg(A)$, $\lambda^d(A)$ is the zero ring (we will try to avoid using this notation in that case).

In general, if $B = \mathrm{End}_A(V)$ and $d \leqslant \mathrm{rdim}(V)$, then $\mathrm{End}_{A^{\otimes d}}(\mathrm{Alt}^d(V))$ is canonically isomorphic to $\lambda^d(B)$.

3.2 The shuffle product

In this section, we give an appropriate generalization of the wedge product on exterior powers of vector spaces, that is to say an associative product from $\mathrm{Alt}^p(V) \otimes_K \mathrm{Alt}^q(V)$ to $\mathrm{Alt}^{p+q}(V)$.

We start by recalling some elementary results about symmetric groups and shuffles. Let us fix a finite totally ordered set X, and a partition $X = \coprod_i I_i$. Then recall that in $\mathfrak{S}_X$ we have the Young subgroup $\mathfrak{S}_{(I_i)}$ and the set of shuffles $Sh_{(I_i)}$. The usefulness of shuffles is explained by the following lemma:

Lemma 3.6. *Any element of $\mathfrak{S}_X$ can be written in a unique way as $\pi\sigma$, with $\pi \in Sh_{(I_i)}$ and $\sigma \in \mathfrak{S}_{(I_i)}$.*

Proof. Let $\tau \in \mathfrak{S}_X$. For any i, we can write $I_i = \{a_{i,1}, \ldots, a_{i,d_i}\}$ such that $a_{i,1} < \cdots < a_{i,d_i}$, but also $I_i = \{b_{i,1}, \ldots, b_{i,d_i}\}$ such that $\tau(b_{i,1}) < \cdots < \tau(b_{i,d_i})$. Then we define a permutation σ_i of I_i by $\sigma_i(b_{i,j}) = a_{i,j}$, and since we do it for every i we get a permutation $\sigma \in \mathfrak{S}_{(I_i)}$.

Let $\pi = \tau\sigma^{-1}$; by construction $\pi \in Sh_{(I_i)}$ since $\pi(a_{i,j}) = \tau(b_{i,j})$ so $\pi(a_{i,1}) < \cdots < \pi(a_{i,d_i})$. The way we defined the σ_i and π makes it clear that this is the only possible decomposition. □

Shuffles also have a nice compatibility with refinements of partitions:

Lemma 3.7. *Suppose that each I_i is itself partitioned as $I_i = \coprod_j J_{i,j}$. Let $\tau \in \mathfrak{S}_X$, and let $\tau = \pi\sigma$ be the decomposition given by lemma 3.6, with σ corresponding to $(\sigma_i) \in \prod_i \mathfrak{S}_{I_i}$. Then τ is a $(J_{i,j})_{i,j}$-shuffle if and only if each σ_i is a $(J_{i,j})_j$-shuffle.*

Proof. Since π is increasing on each I_i, it is clear that τ is increasing on $J_{i,j}$ if and only σ_i is. □

Let us then generalize a little the construction of the anti-symmetrizer of last section: if $A \subset \mathfrak{S}_X$ is any subset, we define

$$alt(A) = \sum_{g \in A} (-1)^g g \in K[\mathfrak{S}_X],$$

so that $s_X = alt(\mathfrak{S}_X)$.

We can record some very basic observations on this construction:

- if $A, B, C \subset \mathfrak{S}_X$ are such that any element of A can be written uniquely as a product of an element of B and an element of C, then $alt(A) = alt(B)alt(C)$;
- if $A_i \subset \mathfrak{S}_{I_i}$ for each i, then $alt(\prod_i A_i) = \bigotimes_i alt(A_i)$.

If we now define the shuffle element $sh_{(I_i)} \in K[\mathfrak{S}_X]$ by

$$sh_{(I_i)} = alt(Sh_{(I_i)}) = \sum_{\pi \in Sh_{(I_i)}} (-1)^\pi \pi, \tag{8}$$

we get the following consequences:

Corollary 3.8. *It holds in $K[\mathfrak{S}_X]$ that*

$$s_X = sh_{(I_i)} \cdot (s_{I_1} \otimes \cdots \otimes s_{I_r})$$

and if each I_i is further partitioned as $I_i = \coprod_j J_{i,j}$, that

$$sh_{(J_{i,j})_{i,j}} = sh_{(I_i)} \cdot (sh_{(J_{1,j})_j} \otimes \cdots \otimes sh_{(J_{r,j})_j}).$$

Proof. The first equality is a corollary of lemma 3.6 using the first observation above, and the second is a consequence of lemma 3.7 using both observations. □

When $X = \{1, \ldots, d\}$ and the partition comes from a decomposition $d = d_1 + \cdots + d_r$, we simply write $sh_{d_1,\ldots,d_r} \in K[\mathfrak{S}_d]$ for the corresponding shuffle element.

This leads to the following definition:

Definition 3.9. *Let A be a central simple algebra over K, and let V be a right A-module. The shuffle algebra of V is defined as the K-vector space*

$$Sh(V) = \bigoplus_{d\in\mathbb{N}} V^{\otimes d}$$

(which is the same underlying space as the tensor algebra $T(V)$ over K) with the product $V^{\otimes p} \otimes_K V^{\otimes q} \to V^{\otimes p+q}$ defined by

$$x\#y = sh_{p,q}(x \otimes y), \tag{9}$$

which we call the shuffle product.

The term algebra is fully justified by the following proposition:

Proposition 3.10. *Let A be a central simple algebra over K, and let V be a right A-module. The shuffle algebra $Sh(V)$ is a $\mathbb{Z}$-graded associative K-algebra with unit $1 \in K = V^{\otimes 0}$.*

Furthermore,

$$\mathrm{Alt}(V) = \bigoplus_{d=0}^{\mathrm{rdim}(V)} \mathrm{Alt}^d(V) \subset Sh(V)$$

is a subalgebra, and is actually the K-subalgebra generated by $V = \mathrm{Alt}^1(V)$. Precisely, if $x \in V^{\otimes p}$ and $y \in V^{\otimes q}$ with $p + q = d$, we have

$$(s_p x)\#(s_q y) = s_d(x \otimes y)$$

and in particular if $x_1, \dots, x_d \in V$:

$$x_1\#\cdots\#x_d = s_d(x_1 \otimes \cdots \otimes x_d).$$

Proof. For the associativity, we make use of corollary 3.8: if $p + q + r = d$, we have

$$sh_{p,q,r} = sh_{p+q,r} \cdot (sh_{p,q} \otimes 1) = sh_{p,q+r} \cdot (1 \otimes sh_{q,r})$$

which by definition of the shuffle product implies that if $x \in V^{\otimes p}$, $y \in V^{\otimes q}$ and $z \in V^{\otimes r}$, $(x\#y)\#z = x\#(y\#z)$. The claims about the grading and the unit are trivial.

The rest of the statement follows directly from the formula $(s_p x)\#(s_q y) = s_d(x\otimes y)$, which is a clear consequence of corollary 3.8 since it implies $sh_{p,q}(s_p \otimes s_q) = s_d$. □

From what we already observed previously, when $A = K$ the alternating algebra $\mathrm{Alt}(V)$ is canonically isomorphic to the exterior algebra $\Lambda(V)$, and the shuffle product corresponds to the wedge product.

One of the main properties of the wedge product is its anti-commutativity, and we do get some version of that property:

Proposition 3.11. *Let A be a central simple algebra over K, and let V be a right A-module. For any $d \in \mathbb{N}$, $x_1, \dots, x_d \in V$ and $\pi \in \mathfrak{S}_d$, we have*

$$x_{\pi(1)}\#\dots\#x_{\pi(d)} = (-1)^\pi (x_1\#\dots\#x_d)\pi.$$

Proof. From proposition 3.1 we see that $x_{\pi(1)}\#\dots\#x_{\pi(d)}$ is $s_d\pi^{-1}(x_1\otimes\cdots\otimes x_d)\pi$, so we can conclude using $s_dg=(-1)^g s_d$ for any $g\in\mathfrak{S}_d$. ☐

We now establish the analogue of the well-known addition formula for exterior powers of vector spaces.

Proposition 3.12. *Let A be a central simple algebra over K, and let U and V be right A-modules. Then for any $d\in\mathbb{N}$ the shuffle product induces an isomorphism of $A^{\otimes d}$-modules :*

$$\bigoplus_{p+q=d}\mathrm{Alt}^p(U)\otimes_K\mathrm{Alt}^q(V)\xrightarrow{\sim}\mathrm{Alt}^d(U\oplus V).$$

We first need the following lemma, which explains how to handle the action of the symmetric groups when two different modules are involved:

Lemma 3.13. *For any $p,q\in\mathbb{N}$ with $p+q=d$, there are two natural structures of left $K[\mathfrak{S}_{p,q}]$-module on $U^{\otimes p}\otimes_K V^{\otimes q}$: the one induced by the left $K[\mathfrak{S}_p]$-module $U^{\otimes p}$ and the left $K[\mathfrak{S}_q]$-module $V^{\otimes q}$; and the restriction of the left $K[\mathfrak{S}_d]$-module structure on $(U\oplus V)^{\otimes d}$.*

Those two structures are actually the same.

Proof. We can reduce by scalar extension to the case where A is split, and by Morita equivalence to $A=K$, in which case the result is about the classical permutation actions, and is therefore clear. ☐

Proof of the Proposition. Lemma 3.13 ensures that we can compute all shuffle products in $\mathrm{Alt}(U\oplus V)$ with no worry ($\mathrm{Alt}(U)$ and $\mathrm{Alt}(V)$ are subalgebras).

Using proposition 3.10, we easily establish that $\mathrm{Alt}^d(U\oplus V)$ is linearly spanned by the elements of the type $x_1\#\cdots\#x_d$ with x_i in U or V. Now using proposition 3.11, we can permute the x_i so that $x_1,\dots,x_p\in U$ and $x_{p+1},\dots,x_d\in V$, at the cost of multiplying on the right by some $\pi\in\mathfrak{S}_d$. But any element of this type is obviously in the image of the map described in the statement of the proposition, so this map is surjective. We may then conclude that it is an isomorphism by checking the dimensions over K (using proposition 3.4 for instance). ☐

Note that in particular this defines a natural $\mathbb{Z}$-graded K-linear isomorphism between $\mathrm{Alt}(U)\otimes_K\mathrm{Alt}(V)$ and $\mathrm{Alt}(U\otimes V)$, but it is not quite an algebra isomorphism.

3.3 Alternating powers of a ε-hermitian form

Now if V is a non-zero A-module equipped with a ε-hermitian form h with respect to some involution σ on A, we want to endow $\mathrm{Alt}^d(V)$ with an induced form $\mathrm{Alt}^d(h)$ such that in the split case we recover the exterior power of the bilinear form. This requires understanding the interaction between the action of the symmetric group and the involutions on the algebras.

Recall that any group algebra $K[G]$ has a canonical involution S given by $S:g\mapsto g^{-1}$, and that if (R,σ) is any ring with involution, its isometry group $\mathrm{Iso}(R,\sigma)$ is the set of $x\in R$ such that $x\sigma(x)=1$ (it is a subgroup of $R^\times$). In particular, G is a subgroup of $\mathrm{Iso}(K[G],S)$.

Proposition 3.14. *Let (A, σ) be an algebra with involution over K. Then for any $d \in \mathbb{N}$ the canonical K-algebra morphism $K[\mathfrak{S}_d] \to A^{\otimes d}$ is a morphism of involutive algebras*

$$(K[\mathfrak{S}_d], S) \to (A^{\otimes d}, \sigma^{\otimes d}).$$

Equivalently, the canonical group morphism $\mathfrak{S}_d \to A^{\otimes d}$ actually takes values in the isometry group $\mathrm{Iso}(A^{\otimes d}, \sigma^{\otimes d})$.

Proof. The equivalence of the two formulations is clear given the form of the canonical involution on $K[\mathfrak{S}_d]$. We can then reduce to the case of $d = 2$ and a transposition, which means we have to prove that the Goldman element is symmetric for σ^2. By definition of g_A, this amounts to the fact that if $g_A = \sum_i a_i \otimes b_i$ with $a_i, b_i \in A$, then for any $x \in A$, $\sum_i \sigma(a_i)x\sigma(b_i) = \mathrm{Trd}_A(x)$. But notice that element is $\sigma(\sum_i b_i\sigma(x)a_i)$, which is $\mathrm{Trd}_A(\sigma(x)) = \mathrm{Trd}_A(x)$ because $\sum_i b_i \otimes a_i = g_A(\sum_i a_i \otimes b_i)g_A = g_A$. □

Corollary 3.15. *Let (A, σ) be an algebra with involution over K, and let (V, h) be an ε-hermitian module over (A, σ). Then for any $d \in \mathbb{N}$, and any $x, y \in V^{\otimes d}$, and any $\theta \in K[\mathfrak{S}_d]$ we have*

$$h^{\otimes d}(\theta \cdot x, y) = h^{\otimes d}(x, S(\theta) \cdot y).$$

In particular, since $S(s_d) = s_d$, we get $h^{\otimes d}(x, s_d y) = h^{\otimes d}(s_d x, y)$.

Proof. Let $B = \mathrm{End}_A(V)$ and $\tau = \sigma_h$, and write $\theta_B \in B^{\otimes d}$ for the image of θ by the canonical morphism. Then proposition 3.14 shows that $\tau^{\otimes d}(\theta_B)$ is the image of $S(\theta)$ in $B^{\otimes d}$, which shows the first formula by definition of the adjoint involution. The fact that $S(s_d) = s_d$ is clear since $g \mapsto g^{-1}$ is bijective on $\mathfrak{S}_d$ and preserves $(-1)^g$. □

This observation allows the following definition:

Definition 3.16. *Let (A, σ) be an algebra with involution over K, and let (V, h) be an ε-hermitian module over (A, σ). We set:*

$$\begin{array}{rrcl} \mathrm{Alt}^d(h) : & \mathrm{Alt}^d(V) \times \mathrm{Alt}^d(V) & \longrightarrow & A^{\otimes d} \\ & (s_d x, s_d y) & \longmapsto & h^{\otimes d}(s_d x, y) = h^{\otimes d}(x, s_d y). \end{array}$$

This is well-defined according to corollary 3.15, since $h^{\otimes d}(s_d x, y)$ only depends on $s_d x$ and not the full x, and conversely $h^{\otimes d}(x, s_d y)$ only depends on $s_d y$.

The definition can be rephrased as

$$\mathrm{Alt}^d(x_1 \# \dots \# x_d, y_1 \# \dots \# y_d) = h^{\otimes d}(x_1 \# \dots \# x_d, y_1 \otimes \dots \otimes y_d). \tag{10}$$

Proposition 3.17. *Let (A, σ) be an algebra with involution over K, and let (V, h) be an ε-hermitian module over (A, σ). The application $\mathrm{Alt}^d(h)$ is an ε^d-hermitian form over $(A^{\otimes d}, \sigma^{\otimes d})$.*

Proof. We have for all $x, y \in V^{\otimes d}$ and all $a, b \in A^{\otimes d}$:

$$\begin{aligned} \mathrm{Alt}^d(h)(s_d x \cdot a, s_d y \cdot b) &= h^{\otimes d}(xa, s_d y b) \\ &= \sigma^{\otimes d}(a) h^{\otimes d}(x, s_d y) b \\ &= \sigma^{\otimes d}(a) \, \mathrm{Alt}^d(h)(s_d x, s_d y) b \end{aligned}$$

and

$$\begin{aligned}\mathrm{Alt}^d(h)(s_d y, s_d x) &= h^{\otimes d}(y, s_d x) \\ &= \varepsilon^d \sigma^{\otimes d}(h^{\otimes d}(s_d x, y)) \\ &= \varepsilon^d \sigma^{\otimes d}(\mathrm{Alt}^d(h)(s_d x, s_d y)). \qquad \square\end{aligned}$$

Example 3.18. When $A = K$ and h is a bilinear form on the vector space V, then if $x = u_1 \otimes \cdots \otimes u_d$ and $y = v_1 \otimes \cdots \otimes v_d$, we get

$$\mathrm{Alt}^d(h)(s_d x, s_d y) = \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \prod_i h(u_{\pi^{-1}(i)}, v_i) = \det(h(u_i, v_j))$$

so when we identify $\mathrm{Alt}^d(V)$ and $\Lambda^d(V)$, $\mathrm{Alt}^d(h)$ does correspond to $\lambda^d(h)$.

Proposition 3.19. *Let (A,σ) be an algebra with involution over K, and let (V,h) be an ε-hermitian module over (A,σ). For any $d \in \mathbb{N}$, we can restrict the ε^d-hermitian form $h^{\otimes d}$ to $\mathrm{Alt}^d(V) \subset V^{\otimes d}$, and we get*

$$h^{\otimes d}_{|\,\mathrm{Alt}^d(V)} = \langle d! \rangle \,\mathrm{Alt}^d(h).$$

Proof. Since s_d is symmetric, we have $h^{\otimes d}(s_d x, s_d y) = h^{\otimes d}(x, (s_d)^2 y)$. But it is easy to see that $s_d^2 = (d!)s_d$, which concludes. $\square$

Note that this means that we could have simply defined $\mathrm{Alt}^d(h)$ in terms of the restriction of $h^{\otimes d}$ in characteristic 0, but in arbitrary characteristic this does not work.

We can then show the compatibility of this construction with the sum formula:

Proposition 3.20. *Let (A,σ) be an algebra with involution over K, and let (U,h) and (V,h') be ε-hermitian modules over (A,σ). The module isomorphism in proposition 3.12 induces an isometry*

$$\bigoplus_{p+q=d} \mathrm{Alt}^p(h) \otimes_K \mathrm{Alt}^q(h') \xrightarrow{\sim} \mathrm{Alt}^d(h \perp h').$$

Proof. Let $u, u' \in U^{\otimes p}$ and $v, v' \in V^{\otimes q}$. Then

$$\begin{aligned}&\mathrm{Alt}^d(h \perp h')((s_p u)\#(s_q v), (s_p u)\#(s_q v)) \\ =&\,\mathrm{Alt}^d(h \perp h')(s_d(u \otimes v), s_d(u' \otimes v')) \\ =&(h \perp h')^{\otimes d}(s_d(u \otimes v), u' \otimes v') \\ =&\sum_{\pi \in \mathfrak{S}_d} (-1)^\pi (h \perp h')^{\otimes d}(\pi(u \otimes v), u' \otimes v'),\end{aligned}$$

where we used proposition 3.10 for the first equality. We want to show that $(h \perp h')^{\otimes d}(\pi(u \otimes v), u' \otimes v') = 0$ if $\pi \notin \mathfrak{S}_{p,q}$. But if $u = x_1 \otimes \cdots \otimes x_p$, $u' = y_1 \otimes \cdots \otimes y_p$, and $v = x_{p+1} \otimes \cdots \otimes x_d$, $v' = y_{p+1} \otimes \cdots \otimes y_d$, then using proposition 3.1:

$$\begin{aligned}&(h \perp h')^{\otimes d}(\pi(u \otimes v), u' \otimes v') \\ &= (h \perp h')^{\otimes d}((x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(d)})\pi, (y_1 \otimes \cdots \otimes y_d)) \\ &= \pi^{-1}(h \perp h')(x_{\pi^{-1}(1)}, y_1) \otimes \cdots \otimes (h \perp h')(x_{\pi^{-1}(d)}, y_d)\end{aligned}$$

which is indeed zero if $\pi \notin \mathfrak{S}_{p,q}$ since at least one of the $(h \perp h')(x_{\pi^{-1}(i)}, y_i)$ will be zero. Hence:

$$\begin{aligned}
&\mathrm{Alt}^d(h \perp h')((s_p u)\#(s_q v), (s_p u)\#(s_q v)) \\
&= \sum_{\pi \in \mathfrak{S}_{p,q}} (-1)^{\pi} (h \perp h')^{\otimes d}(\pi(u \otimes v), u' \otimes v') \\
&= \sum_{\pi_1 \in \mathfrak{S}_p} \sum_{\pi_2 \in \mathfrak{S}_q} (-1)^{\pi_1 \pi_2} (h \perp h')^{\otimes d}(\pi_1 u \otimes \pi_2 v), u' \otimes v') \\
&= h(s_p u, u') \otimes h'(s_q v, v').
\end{aligned}$$

□

Remark 3.21. If $d \leqslant \deg(A)$, then the hermitian form $\mathrm{Alt}^d(\langle 1 \rangle_\sigma)$ induces an adjoint involution $\sigma^{\wedge d}$ on $\lambda^d(A)$. This is essentially the same definition of $\sigma^{\wedge d}$ as in [10] (and it is indeed the same involution).

In general, if $B = \mathrm{End}_A(V)$ and $d \leqslant \mathrm{rdim}(V)$, then the adjoint involution of $\mathrm{Alt}^d(h)$, defined on $\lambda^d(B)$, is $\sigma_h^{\wedge d}$.

3.4 The pre-λ-ring structures

We want to use the previous constructions and results to get a Γ-structure on $\widetilde{GW}(A,\sigma)$. They first translate to a structure on $\widehat{GW}(A,\sigma)$.

Theorem 3.22. *Let (A,σ) be an algebra with involution over K. Then the operation*

$$\mathrm{Alt}^d : SW_\varepsilon(A^{\otimes n}, \sigma^{\otimes n}) \to \{0\} \cup SW_{\varepsilon^d}(A^{\otimes nd}, \sigma^{\otimes nd})$$

extends uniquely to a Γ'-graded pre-λ-structure on $\widehat{GW}(A,\sigma)$. Also $\widehat{\mathrm{rdim}}$ is a graded augmentation, and $\widehat{SW}(A,\sigma)$ is a homogeneous positive structure, making $\widehat{GW}(A,\sigma)$ into a Γ'-structured ring.

Furthermore, $\widehat{GW}$ is a functor from $\mathbf{Br}_h(K)$ to the category of Γ'-structured rings.

Proof. Proposition 3.20 shows that $h \mapsto \sum_d \mathrm{Alt}^d(h) t^d$ defines an additive map from $\widehat{SW}(A,\sigma)$ to $\Lambda_{\Gamma'}(\widehat{GW}(A,\sigma))$, and by universal property of Grothendieck groups, this extends uniquely to a group morphism on $\widehat{GW}(A,\sigma)$. Since Alt^1 is the identity, this is exactly a graded pre-λ-ring structure.

Since we already know that $\widehat{\mathrm{rdim}}$ is a graded ring morphism, we need to check that it is a λ-morphism, which is exactly the content of proposition 3.4. The fact that $\widehat{SW}(A,\sigma)$ is a homogeneous positive structure is mostly clear; only condition (v) on line elements in definition **??** is non-trivial. But the elements of dimension 1 in $\widehat{SW}(A,\sigma)$ are exactly the 1-dimensional quadratic forms in $SW_1(A^{\otimes d}, \sigma^{\otimes d})$ when $A^{\otimes d}$ is split, and clearly such elements are quasi-invertible (for instance because they are invertible up to Morita equivalence), and form a saturated submonoid of such elements.

It remains to show the functoriality; let $f : (B,\tau) \to (A,\sigma)$ be a morphism in $\mathbf{Br}_h(K)$. We already know that the induced map f_* on the $\widehat{GW}$ is a graded ring morphism, which preserves the reduced dimension and $\widehat{SW}$; it remains to check that it preserves the λ-operations. So let (V,h) be an ε-hermitian module over $(B^{\otimes n}, \tau^{\otimes n})$, and let $d \in \mathbb{N}$. What we want to prove is then

$$f_*^{\otimes nd}(\mathrm{Alt}^d(h)) = \mathrm{Alt}^d(f_*^{\otimes n}(h)).$$

Replacing f by $f^{\otimes n}$ if necessary, it is enough to treat the case $n = 1$.

Let (U, g) be the hermitian space over (A, σ) associated to f. Then the underlying module on the left-hand side is

$$(s_d V^{\otimes d}) \otimes_{B^{\otimes d}} U^{\otimes d},$$

and on the right-hand side:

$$s_d(V \otimes_B U)^{\otimes d}.$$

There is an obvious bimodule isomorphism between the two, given by

$$(v_1 \# \dots \# v_d) \otimes (u_1 \otimes \cdots \otimes u_d) \mapsto (v_1 \otimes u_1) \# \dots \# (v_d \otimes u_d),$$

and if we look at the definitions of $g^{\otimes d} \circ \mathrm{Alt}^d(h)$ and $\mathrm{Alt}^d(g \circ h)$, we see that we need to prove that for any $u_i, u_i' \in U$ and $v_i, v_i' \in V$,

$$g^{\otimes d}(u_1 \otimes \cdots \otimes u_d, h^{\otimes d}(v_1 \otimes \cdots \otimes v_d, v_1' \# \dots \# v_d')(u_1' \otimes \cdots \otimes u_d'))$$

is equal to

$$(g \circ h)^{\otimes d}((v_1 \otimes u_1) \otimes \cdots \otimes (v_d \otimes u_d), (v_1' \otimes u_1') \# \dots \# (v_d' \otimes u_d')).$$

It is then a straightforward computation, using proposition 3.1, that both expressions are equal to

$$\sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \left[\bigotimes_i g(u_i, h(v_i, v'_{\pi^{-1}(i)}) u'_{\pi^{-1}(i)}) \right] \pi. \qquad \square$$

Remark 3.23. Note that since $\widehat{GW}(A, \sigma)$ is a graded pre-λ-ring, it is in particular an ungraded pre-λ-ring, but $\widehat{SW}(A, \sigma)$ is *not* a positive structure in this context, because the line elements are not invertible (only quasi-invertible).

Just as for the ring structure, we can then transfer this structure to $\widetilde{GW}(A, \sigma)$, which is the ring we are really interested in.

Corollary 3.24. *Let (A, σ) be an algebra with involution over K. There is a unique Γ-graded pre-λ-ring structure on $\widetilde{GW}(A, \sigma)$ such that the canonical map $\widehat{GW}(A, \sigma) \to \widetilde{GW}(A, \sigma)$ is a lax graded λ-morphism. For this structure, $\widetilde{\mathrm{rdim}}$ is a graded augmentation, and $\widetilde{SW}(A, \sigma)$ a homogeneous positive structure, making $\widetilde{GW}(A, \sigma)$ a Γ-structured ring.*

Furthermore, $\widetilde{GW}$ is a functor from $\mathbf{Br}_h(K)$ to the category of Γ-structured rings.

Proof. All statements are straightforward consequences of the corresponding results for $\widehat{GW}(A, \sigma)$ stated in theorem **??**, taking into account that the canonical ring morphism $\widehat{GW}(A, \sigma) \to \widetilde{GW}(A, \sigma)$ is an additive isomorphism on homogeneous components. $\square$

If (V, h) is an ε-hermitian module over (A, σ), it image by the operation λ^d in $\widetilde{GW}(A, \sigma)$ will be denoted

$$(\Lambda^d(V), \lambda^d(h)) \in SW^{\varepsilon^d}(A^{\otimes r}, \sigma^{\otimes r}),$$

where $r \in \{0, 1\}$ has the same parity as d.

Remark 3.25. Note that, unlike $(\mathrm{Alt}^d(V), \mathrm{Alt}^d(h))$, which is a well-defined hermitian module, only the isometry class of $(\Lambda^d(V), \lambda^d(h))$ is well-defined. This is because although $\varphi^{d,r}_{(A,\sigma)}$ is well-defined as a morphism in $\mathbf{Br}_h(K)$, the morphisms in this category are isometry classes of hermitian bimodules. A particular representative of $(\Lambda^d(V), \lambda^d(h))$ can be singled out by choosing a specific representative of $\varphi^{d,r}_{(A,\sigma)}$, for instance by choosing a parenthesizing of the d factors in $V^{\otimes d}$.

This distinction explains why it is often more convenient to prove things in $\widehat{GW}(A,\sigma)$ first, where we can work with actual modules, and transfer the results to $\widetilde{GW}(A,\sigma)$ by Morita equivalence.

Also note that the isomorphism class of $\Lambda^d(V)$ as a bimodule only depends on σ, and not h. Furthermore, if $d > \mathrm{rdim}_A(V)$, then $\Lambda^d(V) = 0$.

Example 3.26. We know from example 2.9 that as a ring $\widetilde{GW}(K, \mathrm{Id}) \simeq GW^{\pm}[\mathbb{Z}/2\mathbb{Z}]$. We also know from proposition 1.5 that $GW^{\pm}[\mathbb{Z}/2\mathbb{Z}]$ has a canonical Γ-graded pre-λ-ring structure, since by example 1.2 $GW^{\pm}(K)$ is a $\mu_2(K)$-graded pre-λ-ring. Then actually $\widetilde{GW}(K, \mathrm{Id}) \simeq GW^{\pm}[\mathbb{Z}/2\mathbb{Z}]$ as Γ-graded pre-λ-rings.

Remark 3.27. If $f : (B,\tau) \to (A,\sigma)$ is a morphism in $\mathbf{Br}_h(K)$, corresponding to the ε-hermitian form h, then by definition $f_*(\langle 1\rangle_\tau) = h$, and since f_* is compatible with the λ-operations, we have $\lambda^d(h) = f_*(\lambda^d(\langle 1\rangle_\tau))$. Thus to be able to compute the exterior powers of any ε-hermitian form, we just need to be able to do the computation in the special case of diagonal forms $\langle 1\rangle_\tau$ for any involution τ.

Since we are mostly interested in even λ-powers in applications, it can be useful to give a more explicit description of (a representative of) $\lambda^{2d}(\langle 1\rangle_\sigma)$ for some (A,σ), which by the preceding remark is enough to describe $\lambda^{2d}(h)$ for any h.

For any $d \in \mathbb{N}$, and any $i \in \{1, \dots, d\}$, let us write $\tau_i \in (\mathbb{Z}/2\mathbb{Z})^d$ for the element with a 1 in the ith position. If (A,σ) is an algebra with involution, let us also write $\sigma_i = (1 \otimes \cdots \otimes \sigma \otimes \cdots \otimes 1)$ with the σ in the ith position also. Then on the K-vector space $A^{\otimes d}$ we get a left action of $(\mathbb{Z}/2\mathbb{Z})^d$ where τ_i acts as $\varepsilon(\sigma)\sigma_i$. Recall that we also have a left and right action of $\mathfrak{S}_d$ coming from the Goldman element, so we get a left action of $\mathfrak{S}_d \times \mathfrak{S}_d$ by $x \mapsto \pi_1 x \pi_2^{-1}$ for all $x \in A^{\otimes d}$ and $\pi_1, \pi_2 \in \mathfrak{S}_d$. Combining those two actions defines an action of the free product $(\mathfrak{S}_d \times \mathfrak{S}_d) * (\mathbb{Z}/2\mathbb{Z})^d$.

Lemma 3.28. *Let (A,σ) be an algebra with involution over K, and let $d \in \mathbb{N}^*$. The action of $(\mathfrak{S}_d \times \mathfrak{S}_d) * (\mathbb{Z}/2\mathbb{Z})^d$ on the K-vector space $A^{\otimes d}$ factors through an action of $\mathfrak{S}_{2d}$, through the morphism $(\mathfrak{S}_d \times \mathfrak{S}_d) * (\mathbb{Z}/2\mathbb{Z})^d \to \mathfrak{S}_{2d}$ sending τ_i to the transposition $(2i-1, 2i)$, and identifying $\mathfrak{S}_d \times \mathfrak{S}_d$ with the Young subgroup of $\mathfrak{S}_{2d}$ corresponding to the partition of $\{1, \dots, d\}$ into odd and even numbers.*

Proof. We may reduce to the split case; in this case, $A \simeq \mathrm{End}_K(U)$ for some K-vector space U, and $\sigma = \sigma_b$ for some ε-symmetric bilinear form on U. Then (see [10, §5.A]) we have a natural identification $A \simeq U \otimes_K U$, such that the product of A becomes $(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = b(y_1, x_2)(x_1 \otimes y_2)$, the reduced trace of $x \otimes y$ is $\varepsilon b(x,y)$, and $\sigma(x \otimes y) = \varepsilon(y \otimes x)$. If (e_i) is any basis of U, and (e_i^*) is its dual basis with respect to b (so $b(e_i^*, e_j) = \delta_{i,j}$), the unit $1_A \in A$ is

$\sum_i e_i \otimes e_i^*$, and the Goldman element $g_A \in A \otimes_K A$ is $\sum_{i,j} e_i \otimes e_j^* \otimes e_j \otimes e_i^*$ (which shows that those expressions actually do not depend on the choice of basis).

It is then an easy exercise to see that on $A \otimes_K A \simeq U_1 \otimes_K U_2 \otimes_K U_3 \otimes_K U_4$, left multiplication by g_A switches U_1 and U_3, and right multiplication by g_A switches U_2 and U_4. All in all, this shows that the action of $(\mathfrak{S}_d \times \mathfrak{S}_d) * (\mathbb{Z}/2\mathbb{Z})^d$ on $A^{\otimes d} \simeq U_1 \otimes \cdots \otimes U_{2d}$ is the permutation action, through the morphism $(\mathfrak{S}_d \times \mathfrak{S}_d) * (\mathbb{Z}/2\mathbb{Z})^d \to \mathfrak{S}_{2d}$ described in the statement. □

Proposition 3.29. *Let (A,σ) be an algebra with involution over K, and let $\varepsilon = \varepsilon(\sigma)$ and $d \in \mathbb{N}^*$. Let $M_{\sigma,d}$ be the subspace of $A^{\otimes d}$ consisting of the elements x such that $\sigma_i(x) = -\varepsilon x$ for all $1 \leqslant i \leqslant d$, and let $N_{\sigma,d} = \mathrm{Alt}^d(A) \cap M_{\sigma,d}$. Then using the structure of $K[\mathfrak{S}_{2d}]$-module on $A^{\otimes d}$ described in lemma 3.28, $N_{\sigma,d} = s_{2d}A^{\otimes d}$.*

A representative of the bilinear space $\lambda^{2d}(\langle 1\rangle_\sigma)$ is given by $(N_{\sigma,d}, b_{\sigma,d})$, where

$$b_{\sigma,d}(s_{2d}x, s_{2d}y) = \varepsilon^d Trd_{A^{\otimes d}}(s_{2d}xy).$$

Furthermore, if the characteristic of K is either 0 or $p > 2d$, then $b_{\sigma,d}$ is isometric to the restriction of $\langle (2d)!\rangle T_\sigma^{\otimes d}$ from $A^{\otimes d}$ to $N_{\sigma,d}$.

Proof. From theorem 2.4, we see that $\varphi^{2d,0}_{(A,\sigma)}$ can be represented by the hermitian bimodule $(|A|_\sigma^{\otimes d}, T_\sigma^{\otimes d})$, where $A^{\otimes 2d}$ acts on the left on $|A|_\sigma^{\otimes d}$ by regrouping $A^{\otimes 2d}$ as $(A \otimes_K A) \otimes \cdots \otimes (A \otimes_K A)$ and using the twisted action of $A \otimes_K A$ on $|A|_\sigma$ (it is not the only possibility, but it is arguably the most natural).

Then, since the left action of the symmetric group is compatible with Morita equivalences, $(\Lambda^{2d}(A), \lambda^{2d}(\langle 1\rangle_\sigma))$ can be obtained from $(|A|_\sigma^{\otimes d}, T_\sigma^{\otimes d})$ (with the induced action of $K[\mathfrak{S}_{2d}]$) exactly the same way that $(\mathrm{Alt}^{2d}(A), \mathrm{Alt}^{2d}(\langle 1\rangle\sigma))$ was obtained from $(A^{\otimes 2d}, \langle 1\rangle_\sigma^{\otimes 2d})$.

It turns out that the induced action of $K[\mathfrak{S}_{2d}]$ on $|A|_\sigma^{\otimes d}$ (which as a vector space is simply $A^{\otimes d}$) is precisely the one described in lemma 3.28.

The fact that $N_{\sigma,d} = s_{2d}A^{\otimes d}$ is now a consequence of lemma 3.2, since $\mathfrak{S}_{2d}$ is generated by $\mathfrak{S}_d \times \{1\}$ and the τ_i, and $\mathrm{Alt}^d(A)$ is (still by lemma 3.2) the subspace of elements x such that $gx = -(-1)^g x$ for all $g \in \mathfrak{S}_d \times \{1\}$.

This shows that $\lambda^{2d}(\langle 1\rangle_\sigma)$ is indeed given by $(N_{\sigma,d}, b_{\sigma,d})$ with the formula for $b_{\sigma,d}$ given in the statement, using that since $\sigma^{\otimes d}(s_{2d}x) = \varepsilon^d s_{2d}x$, $T_\sigma^{\otimes d}(s_{2d}x, y)$ is $\varepsilon^d Trd_{A^{\otimes d}}(s_{2d}xy)$.

The last statement is a direct application of proposition 3.19, since it shows that in any characteristic the restriction of $T_\sigma^{\otimes d}$ to $N_{\sigma,d}$ is $\langle (2d)!\rangle b_{\sigma,d}$. □

We are particularly interested in the case where $2d = 2$:

Corollary 3.30. *Let (A,σ) be an algebra with involution over K. Then $\lambda^2(\langle 1\rangle_\sigma)$ is the isometry class of $\langle 2\rangle T_\sigma^{-\varepsilon(\sigma)}$.*

Proof. We just have to see that $N_{\sigma,1}$ is $\mathrm{Sym}^{-\varepsilon(\sigma)}(A,\sigma)$, so that we can apply the last statement of proposition 3.29 to $d = 1$, since $T_\sigma^{-\varepsilon(\sigma)}$ is precisely the restriction of T_σ to $\mathrm{Sym}^{-\varepsilon(\sigma)}(A,\sigma)$.

But $\mathrm{Alt}^1(A) = A$, and $M_{\sigma,1}$ is $\mathrm{Sym}^{-\varepsilon(\sigma)}(A,\sigma)$ by definition. □

Remark 3.31. A more intrinsic description of $\lambda^2(\langle 1\rangle_\sigma)$, which would also work in characteristic 2, and relies on the first description of $b_{\sigma,1}$ in proposition 3.29, corresponds to the form in [10, Exercise 2.15].

4 Determinants and duality

4.1 The determinant of an involution

In proposition 1.26, we defined a notion of determinant for any G-structured ring, and in example 1.28 we checked that this coincides with the usual notion of determinant for bilinear forms when applied to $GW^{\pm}(K)$ (identifying square classes and 1-dimensional quadratic forms). If we know look at $\widetilde{GW}(A,\sigma)$, this defines a notion of determinant for ε-hermitian forms, and thus of involutions (which is more or less the same).

Definition 4.1. *Let (A,σ) be an algebra with involution over K. The determinant of (A,σ) (also referred to as the determinant of σ) is*

$$\det(A,\sigma) = \det(\sigma) = \det(\langle 1\rangle_\sigma) \in \widetilde{GW}(A,\sigma).$$

By Morita functoriality, this means that in general $\det(h) = \det(\sigma_h)$. Now there is already in the literature a notion of determinant of orthogonal involution on algebras of even degree, see for instance [10, 7.2]. In that case those notions coincide: since the scalar extension to a generic splitting field is injective on the group of square classes, this can be checked when A is split, where it is clear since both definitions generalize the determinant of a quadratic form.

For symplectic involutions (necessarily in even degree), this splitting trick also shows that the determinant is always $\langle 1\rangle$, so there is no meaningful notion of determinant of a symplectic involution, as is well-known.

From those two cases, we can see in particular that if σ is symplectic, then for any invertible $a \in \mathrm{Sym}^{\varepsilon}(A,\sigma)$, $\det(\langle a\rangle_\sigma) = \langle Nrd_A(a)\rangle$. (If σ is not assumed to be symplectic, we rather get $\det(\langle a\rangle_\sigma) = \langle \det(\sigma) Nrd_A(a)\rangle$.)

Now when $\deg(A)$ is odd, something a little different is happening: there is no classical notion of determinant of the involution in that case. This is because, since A must be split and σ orthogonal, σ is adjoint to some symmetric bilinear form b which is only well-defined up to a scalar, and since $\det(\langle\lambda\rangle b) = \langle\lambda\rangle \det(b)$ because the dimension is odd, you cannot attach a square class to that situation. The definition we gave circumvents this issue by defining $\det(\sigma)$ as a line element of $GW(A,\sigma)$. The line elements in $GW(K)$ correspond canonically to square classes, but the line elements in $GW(A,\sigma)$ only correspond to square classes *up to a choice of Morita equivalence*, which exactly compensates the classical obstruction.

This is obviously not a very deep improvement on the previous definitions, but it does offer a natural and uniform definition which covers all cases.

4.2 Determinant duality

We now show that $\widehat{GW}(A,\sigma)$ and $\widetilde{GW}(A,\sigma)$ have a good theory of duality, and in particular satisfy the determinant duality property. We already noted that it was the case for $GW^{\pm}(K)$ in example **??**, but we provided a quick ad how proof using diagonalization. If we want to deduce the general case from the split case, we need to define a *canonical* map which is the candidate for the isometry. This is why it is convenient to first show the result for $\widehat{GW}(A,\sigma)$.

First, we establish duality structures for $\widehat{GW}(A,\sigma)$ and $\widetilde{GW}(A,\sigma)$.

Proposition 4.2.

There in a classical duality, for vector bundles or quadratic forms, involving the determinant: namely, if some element x has dimension n then

$$\lambda^n(x)\lambda^d(x) = \lambda^{n-d}(x)$$

for $0 \leqslant d \leqslant n$. For quadratic forms, this is easily checked on a diagonalization. Indeed, if $q = \langle a_1, \ldots, a_n \rangle$ then $\lambda^d(q)$ has a diagonalization given by the a_I where $I \subset \{1, \ldots, n\}$ has size d, and $a_I = \prod_{i \in I} a_i$, so $\lambda^n(x)\langle a_I \rangle = \langle a_{\overline{I}} \rangle$ shows the formula (where $\overline{I}$ is the complementary of I). For anti-symmetric bilinear forms, it is even more trivial, since they are characterized by their dimension, so the formula amounts to just $\binom{n}{d} = \binom{n}{n-d}$. We wish to show that a similar property holds in $\widetilde{GW}(A, \sigma)$, and we first write general definitions for this property.

For any commutative ring R and any polynomial $P \in R[t]$, let us write $\overline{P} = t^n P(1/t)$ where $n = \deg(P)$; in other words, $\overline{P}$ is the mirror-reversed polynomial of P. Let S_R be the multiplicative submonoid of $R[t]$ consisting of the polynomials with invertible leading and constant coefficients. Then $P \mapsto \overline{P}$ is an involutive endomorphism of this monoid.

Definition 4.3. *Let R be a G-structured ring. A duality on R is an involutive ring endomorphism $x \mapsto x^*$ of R such that:*

- *if $x \in R_g$, then $x^* \in R_{-g}$;*
- $\lambda^d(x^*) = \lambda^d(x)^*$

for any homogeneous positive element $x \in R$, we have

$$\det(x)\lambda_t(x) = \overline{\lambda_t(x)} \in R[t].$$

In the classical setting of vector spaces and bilinear forms, the determinant induces a well-known duality: if V is a vector space of dimension $n = p + q$, the natural pairing

$$\Lambda^p(V) \otimes_K \Lambda^q(V) \longrightarrow \Lambda^n(V)$$

defines a canonical isomorphism

$$\Lambda^p(V) \simeq \mathrm{Hom}_K(\Lambda^q(V), \Lambda^n(V)) \simeq \Lambda^n(V) \otimes_K \Lambda^q(V)^*. \tag{11}$$

Of course since $\Lambda^n(V)$ has dimension 1, this means that $\Lambda^p(V) \approx \Lambda^q(V)^*$ non-canonically, but this is a coincidence, and is not true when we try to generalize to projective modules, or vector bundles.

Now if V is equipped with a non-degenerate ε-symmetric bilinear form b, then using the identification $V \simeq V^*$ provided by b, the above isomorphism gives a natural isometry

$$(\Lambda^p(V), \lambda^p(b)) \simeq (\Lambda^n(V), \lambda^n(b)) \otimes_K (\Lambda^q(V), \lambda^q(b)),$$

where of course $(\Lambda^n(V), \lambda^n(b))$ is the determinant of (V, b), and in particular there is an equality $\lambda^p(b) = \det(b)\lambda^q(b)$ in $GW^{\pm}(K)$.

The goal of this section is to generalize this observation to general ε-hermitian forms.

Theorem 4.4. *Let (A,σ) be an algebra with involution over K, let (V,h) be an ε-hermitian module over (A,σ) of reduced dimension n, and let $p,q \in \mathbb{N}$ such that $p+q=n$. Then there is a canonical isometry*

$$(\Lambda^p(V), \lambda^p(h)) \xrightarrow{\sim} \det(V,h)\cdot(\Lambda^q(V),\lambda^q(h)).$$

This is an isometry of either quadratic spaces (when p is even) or of ε-hermitian forms over (A,σ) (when p is odd). Note that when n and q are odd, there is a slight ambiguity as to what $\det(V,h)\cdot(\Lambda^q(V),\lambda^q(h))$ means; in that case, it is the quadratic space obtained by canonical Morita equivalence from the hermitian module $\det(V,h)\otimes(\Lambda^q(V),\lambda^q(h))$ over $(A^{\otimes 2},\sigma^{\otimes 2})$. This being said, when n is odd A is split, so this does not give a lot more than the classical duality. When either n or q is even, the notation simply designates the tensor product. Obviously, this implies:

Corollary 4.5. *In the conditions of theorem 4.4, we have the following equality in $\widetilde{GW}(A,\sigma)$:*

$$\lambda^p(h) = \det(h)\cdot\lambda^q(h).$$

Similar to the split case discussed above, we can first establish a general duality that does not depend on the presence of any involution or hermitian form, and then specialize it to this case. We first discuss various related notions of dual for our modules.

Let A and B be central simple algebras over K, and let V be a B-A-bimodule such that $B \simeq \mathrm{End}_A(V)$ (so V establishes a Morita equivalence between B and A). Then we can consider three A-B-bimodules that we might want to call "the dual" of V; it turns out they are all canonically isomorphic:

Lemma 4.6. *Consider the three A-B-bimodules $\mathrm{Hom}_A(V,A)$, $\mathrm{Hom}_B(V,B)$ and $\mathrm{Hom}_K(V,K)$, where the actions of A and B on each space are induced by the actions on V. Then there is a canonical commuting triangle of bimodule isomorphisms*

$$\begin{array}{ccc} \mathrm{Hom}_A(V,A) & \longrightarrow & \mathrm{Hom}_B(V,B) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_K(V,K) & \end{array}$$

where the correspondences between $f_A \in \mathrm{Hom}_A(V,A)$, $f_B \in \mathrm{Hom}_B(V,B)$ and $f_K \in \mathrm{Hom}_K(V,K)$ are given by

$$f_B(v)w = v f_A(w), \tag{12}$$

and

$$f_K(v) = \mathrm{Trd}_A(f_A(v)) = \mathrm{Trd}_B(f_B(v)). \tag{13}$$

Proof. It is an easy verification that the maps are well-defined and are bimodule morphisms. To see that they are bijective it suffices to check that they are injective, since the K-dimensions are the same. It is immediate that $f_A = 0$ if and only if $f_B = 0$, and if $\mathrm{Trd}_A(f_A(v)) = 0$ for all $v \in V$, then for any $a \in A$:

$$\mathrm{Trd}_A(f_A(v)a) = \mathrm{Trd}_A(f_A(va)) = 0,$$

so $f_A(v) = 0$ since the trace form is non-degenerate (the reasoning is of course the same for f_B).

It remains to show that the triangle commutes, that is we need to prove that $\mathrm{Trd}_A(f_A(v)) = \mathrm{Trd}_B(f_B(v))$ for all $v \in V$. It is enough to check this in the split case; then $A \simeq \mathrm{End}_K(U) \simeq U \otimes_K U^*$, $B \simeq \mathrm{End}_K(W) \simeq W \otimes_K W^*$, and $V \simeq W \otimes_K U^*$. The reduced trace of A is given by $(u \otimes \varphi \mapsto \varphi(u))$, and likewise for B. We have an identification $V^* \simeq U \otimes_K W^*$, such that if f_A corresponds to $u \otimes \varphi$, then for $x = w \otimes \psi \in V$, $f_A(x) = \psi(u) w \otimes \varphi$ and $f_B(x) = \varphi(w) u \otimes \psi$. In the end:

$$\mathrm{Trd}_B(f_B(x)) = \psi(u)\varphi(w) = \mathrm{Trd}_A(f_A(x)). \qquad \square$$

We then use V^* to refer indifferently to any of these bimodules. This also naturally defines a B^{op}-A^{op}-bimodule, which we denote by V^{-1}. The determinant duality for vector spaces uses a canonical correspondence between maps $U \otimes V \to W$ and maps $U \to W \otimes V^*$. We now give more general statements, which are valid over non-commutative rings and are compatible with Morita equivalences:

Proposition 4.7. *Let R be a commutative ring, and let A and B be R-algebras. Let U, V and W be right modules over respectively A, B and $A \otimes_R B$. We endow B with its standard structure of right $(B \otimes_K B^{op})$-module. There is a canonical bijective correspondence between morphisms of $A \otimes_K B$-modules*

$$U \otimes_R V \longrightarrow W$$

and morphisms of $(A \otimes_R B \otimes_R B^{op})$-modules

$$U \otimes_K B \longrightarrow \mathrm{Hom}_R(V, W).$$

Furthermore, given Morita equivalences over R between A and B and some R-algebras D and E respectively, if U', V' and W' are the modules corresponding to U, V and W respectively, then the equivalences induce a commutative square of R-linear isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{A\otimes_R B}(U \otimes_R V, W) & \longrightarrow & \mathrm{Hom}_{A\otimes_R B\otimes_R B^{op}}(U \otimes_R B, \mathrm{Hom}_R(V, W)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D\otimes_R E}(U' \otimes_R V', W') & \longrightarrow & \mathrm{Hom}_{D\otimes_R E\otimes_R E^{op}}(U' \otimes_R E, \mathrm{Hom}_R(V', W')). \end{array}$$

Proof. For the first part of the statement, to each $f \in \mathrm{Hom}_{A\otimes_R B}(U \otimes_R V, W)$ we associate $g : U \otimes_R B \to \mathrm{Hom}_R(V, W)$ defined by

$$u \otimes b \mapsto (v \mapsto f(u \otimes vb)),$$

and conversely to each such g we associate

$$u \otimes v \mapsto g(u \otimes 1)(v).$$

It is straightforward to check that these associations are well-defined, R-linear, and inverse of each other.

Now say the equivalence between A and D (resp. B and E) is given by the A-D-bimodule M (resp. the B-E-bimodule N). Let us start with $f \in$

$\mathrm{Hom}_{A\otimes_R B}(U \otimes_R V, W)$. It induces $f' : U' \otimes_R V' \to W'$ by tensoring on both sides by $M \otimes_R N$ over $A \otimes_R B$, given that by construction $U' = U \otimes_A M$, $V' = V \otimes_B N$, and $W' = W \otimes_{A\otimes B} (M \otimes_R N)$.

Let $g : U \otimes_R B \to \mathrm{Hom}_R(V, W)$ be the morphism associated to f, and $g' : U' \otimes_R E \to \mathrm{Hom}_R(V', W')$ the morphism induced by Morita equivalence. We need to check that g' is the morphism associated to f' by the method above, which results from a long but straightforward verification. □

Corollary 4.8. *Now let again $R = K$, and A, B, D and E be central simple algebras over K. Then the statements of proposition 4.7 hold when replacing $\mathrm{Hom}_R(V, W)$ (resp. $\mathrm{Hom}_R(V', W')$) with $W \otimes_K V^{-1}$ (resp. $W' \otimes_K (V')^{-1}$). That is, we get a commutative square of K-linear isomorphisms*

$$\begin{array}{ccc} \mathrm{Hom}_{A\otimes_K B}(U \otimes_K V, W) & \longrightarrow & \mathrm{Hom}_{A\otimes_K B\otimes_K B^{op}}(U \otimes_R B, W \otimes_K V^{-1}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D\otimes_K E}(U' \otimes_K V', W') & \longrightarrow & \mathrm{Hom}_{D\otimes_K E\otimes_K E^{op}}(U' \otimes_K E, W' \otimes_K (V')^{-1}). \end{array}$$

Proof. This follows from the fact that $W \otimes_K V^{-1}$ is isomorphic to $\mathrm{Hom}_K(V, W)$ as a $(A \otimes_K B \otimes_K B^{op})$-module, where $A \otimes_K B$ acts on $\mathrm{Hom}_K(V, W)$ through W, and B^{op} acts through V. Indeed, $\mathrm{Hom}_K(V, W) \simeq W \otimes_K \mathrm{Hom}_K(V, K)$ as vector spaces, and it is easy to check that the module structures correspond.

We also need to verify that this isomorphism is compatible with the Morita equivalences, which is just a straightforward unfolding of the definitions. □

We can now state our first version of the determinant duality:

Proposition 4.9. *Let A be a central simple algebra over K, let V be a right A-module of reduced dimension n, and let $p, q \in \mathbb{N}$ such that $p + q = n$. If we consider $A^{\otimes q}$ as a right $(A^{\otimes q} \otimes_K (A^{\otimes q})^{op})$-module in the standard way, there is a canonical isomorphism of right $(A^{\otimes n} \otimes_K (A^{\otimes q})^{op})$-modules*

$$\mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^{-1}.$$

Furthermore, this isomorphism is compatible with Morita equivalences.

Proof. If we apply the correspondance of corollary 4.8 to the shuffle map

$$\mathrm{Alt}^p(V) \otimes_K \mathrm{Alt}^q(V) \longrightarrow \mathrm{Alt}^n(V),$$

we get our canonical morphism of $(A^{\otimes n} \otimes_K (A^{\otimes q})^{op})$-modules

$$\mathrm{Alt}^p(V) \otimes_K A^{\otimes q} \longrightarrow \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)^{-1}.$$

The compatibility with Morita equivalences follows from the corresponding statement in corollary 4.8, and the fact that the Alt^p construction is also compatible with Morita equivalences, which we show in the proof of theorem **??**.

To prove that this is an isomorphism, it is enough to check it after extending the scalars to a splitting field of A and B. But since it is compatible with Morita equivalence, we can actually reduce to the case $A = B = K$, in which case it is simply the classical isomorphism $\Lambda^p(V) \xrightarrow{\sim} \Lambda^n(V) \otimes_K (\Lambda^q(V))^*$. □

Remark 4.10. The module isomorphism $\Phi_{p,q}(V)$ induces an isomorphism between the endomorphism algebras of either side: this gives a canonical isomorphism $\lambda^p(A) \simeq \lambda^q(A)^{op}$. This is the isomorphism alluded to in [10, exercise II.12]. In particular, when $n = 2m$, this defines an isomorphism $\lambda^m(A) \simeq \lambda^m(A)^{op}$ which corresponds to the so-called canonical involution on $\lambda^m(A)$ (see [10, §10.B]).

If there is an involution σ on A, then the isomorphism of proposition 4.9 can naturally be turned into a an isomorphism of $A^{\otimes n+q}$-modules, by twisting the action of $(A^{\otimes q})^{op}$ through $\sigma^{\otimes q}$. To limit the confusion regarding the various "natural" actions, let us write $M_q(A,\sigma)$ for $A^{\otimes q}$ seen as an $A^{\otimes 2q}$-module (on the right) this way, meaning that if $x \in M_q(A,\sigma)$ and $a,b \in A^{\otimes q}$, $x \cdot (a \otimes b) = \sigma^{\otimes q}(a)xb$. Thus after this twisting, on the left-hand side of the isomorphism we get $\mathrm{Alt}^p(V) \otimes_K M_q(A,\sigma)$.

If in addition V carries an ε-hermitian form h, then using the canonical map $\widehat{\mathrm{Alt}^q(h)} : \mathrm{Alt}^q(V)^{-1} \to \mathrm{Alt}^q(V)$, the right-hand side becomes $\mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V)$, so that from the proposition we obtain a natural isomorphism of $A^{\otimes n+q}$-modules

$$\mathrm{Alt}^p(V) \otimes_K M_q(A,\sigma) \xrightarrow{\sim} \mathrm{Alt}^n(V) \otimes_K \mathrm{Alt}^q(V). \tag{14}$$

Note that $M_q(A,\sigma)$ carries a natural hermitian form over $(A^{\otimes 2q}, \sigma^{\otimes 2q})$, which we will write $\omega_q : M_q(A,\sigma) \times M_q(A,\sigma) \to A^{\otimes 2q}$, given by

$$\omega_q(x,y) = (x \otimes 1) \cdot \overline{g_{A^{\otimes q}}} \cdot (1 \otimes y)$$

where $\overline{g_{A^{\otimes q}}}$ is the "twisted goldman element" $(\mathrm{Id} \otimes \sigma^{\otimes q})(g_{A^{\otimes q}}) \in A^{\otimes 2q}$.

Lemma 4.11. *The application ω_q is indeed a hermitian form on $M_q(A,\sigma)$, and defines the inverse hermitian Morita equivalence to $\varphi^{2q,0}_{(A,\sigma)}$ described in theorem 2.4.*

Proof. From theorem 2.4 and the description of $\varphi^{2,0}_{(A,\sigma)}$ which follows it, $\varphi^{2q,0}_{(A,\sigma)}$ is given by the bilinear form $(x,y) \mapsto \mathrm{Trd}_{A^{\otimes q}}(\sigma^{\otimes q}(x)y)$ on $A^{\otimes q}$, which is given the twisted $A^{\otimes 2q}$-action on the left.

From the explicit description of the inverse of a hermitian Morita equivalence (see from instance [9]), we need to check that if $x,y,z \in A^{\otimes q}$, then

$$\omega_q(x,y) \cdot z = x \, \mathrm{Trd}_{A^{\otimes q}}(\sigma^{\otimes q}(y)z).$$

This then follows from the definition of ω_q and the fact that, by the definition of the Goldman element, $\overline{g_{A^{\otimes q}}} \cdot x = \mathrm{Trd}_{A^{\otimes q}}(x)$ for any $x \in A^{\otimes q}$. □

This gives the hermitian version of the determinant duality:

Proposition 4.12. *Let (A,σ) be an algebra with involution over K, let (V,h) be a ε-hermitian module over (A,σ) of reduced dimension n, and let $p,q \in \mathbb{N}$ such that $p+q=n$. Then the $A^{\otimes n+q}$-module isomorphism (14) is an isometry*

$$\mathrm{Alt}^p(h) \otimes \omega_q \xrightarrow{\sim} \mathrm{Alt}^n(h) \otimes \mathrm{Alt}^q(h),$$

which is compatible with hermitian Morita equivalences.

Proof. The fact that the map is compatible with hermitian Morita equivalences follows from the construction and the corresponding statement in proposition 4.9. Only the fact that ω_q is well-behaved with respect to hermitian Morita equivalences may pose a difficulty, but it follows either from a direct computation from its definition, or from its caracterisation in lemma 4.11, given that the $\varphi^{2q,0}_{(A,\sigma)}$ are themselves compatible with hermitian Morita equivalences, as stated in 2.4.

We now only have to check that the map is an isometry; this can be done after extending the scalars to a splitting field, so we may assume that A is split. The compatibility with hermitian Morita equivalences then reduces to the case where $(A, \sigma) = (K, \mathrm{Id})$. In that case, we are left with the classical isometry $\lambda^p(b) \simeq \lambda^n(b) \otimes \lambda^q(b)$ where b is a bilinear form. □

The theorem 4.4 is now an easy consequence:

Proof of Theorem 4.4. The isometry in 4.12 is compatible with hermitian Morita equivalences, and the statement in the theorem is exactly obtained from the one in the proposition by applying the canonical hermitian Morita equivalence $\varphi^{n+q,i}_{(A,\sigma)}$ (where i is the remainder mod 2 of $n + q$). Indeed, this follows from the definition of the $\lambda^d(h)$, and from lemma 4.11 for the disappearance of ω_q. □

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